MA 226 April 13, 2005

More on undamped sinusoidal forcing

Last class we considered the following example.

Example.

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t$$

For $\omega \neq \pm\sqrt{3}$, we obtained the general solution

$$y(t) = k_1 \cos \sqrt{3}\, t + k_2 \sin \sqrt{3}\, t + \frac{1}{3-\omega^2} \cos \omega t,$$

and we concentrated on the initial condition $(y(0), y'(0)) = (0, 0)$. The solution to this initial-value problem is

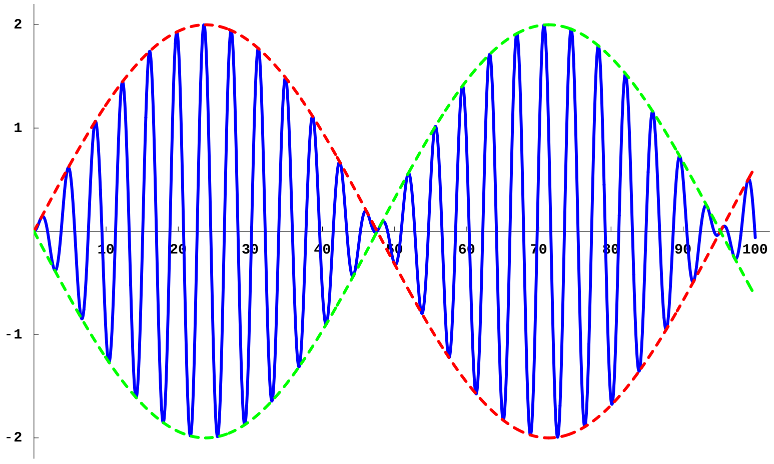
$$y(t) = \frac{1}{3-\omega^2} \left(\cos \omega t - \cos \sqrt{3}\, t \right).$$

We used a trig identity to turn this difference between two cosine functions into a product of sine functions.

Example.

$$\cos 1.6t - \cos \sqrt{3}\, t = 2(\sin 1.67t)(\sin 0.066t)$$

The first sine function has a period of 3.76 and the second sine function has a period of 95.2. Here is the resulting graph:



Let's return to the solution to

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t$$

that satisfies the initial condition $(y(0), y'(0)) = (0, 0)$. If $\omega \neq \pm\sqrt{3}$, the solution is

$$y(t) = \frac{1}{3-\omega^2}(\cos \omega t - \cos \sqrt{3}\, t).$$

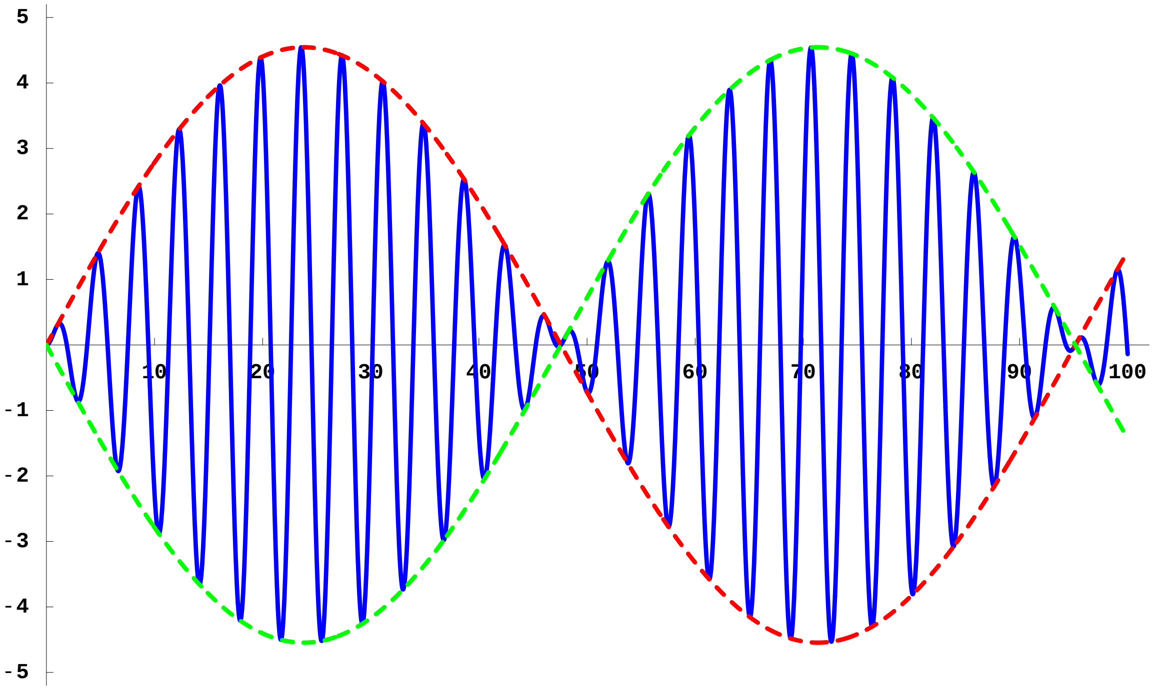
Applying the trig identity, we obtain

$$y(t) = \frac{-2}{3-\omega^2}\ (\sin \alpha t)\ (\sin \beta t)$$

where

$$\alpha = \frac{\omega + \sqrt{3}}{2} \quad \text{and} \quad \beta = \frac{\omega - \sqrt{3}}{2}.$$

Here is the graph of this solution in the case where $\omega = 1.6$.



What happens if $\omega = \sqrt{3}$?

Example.

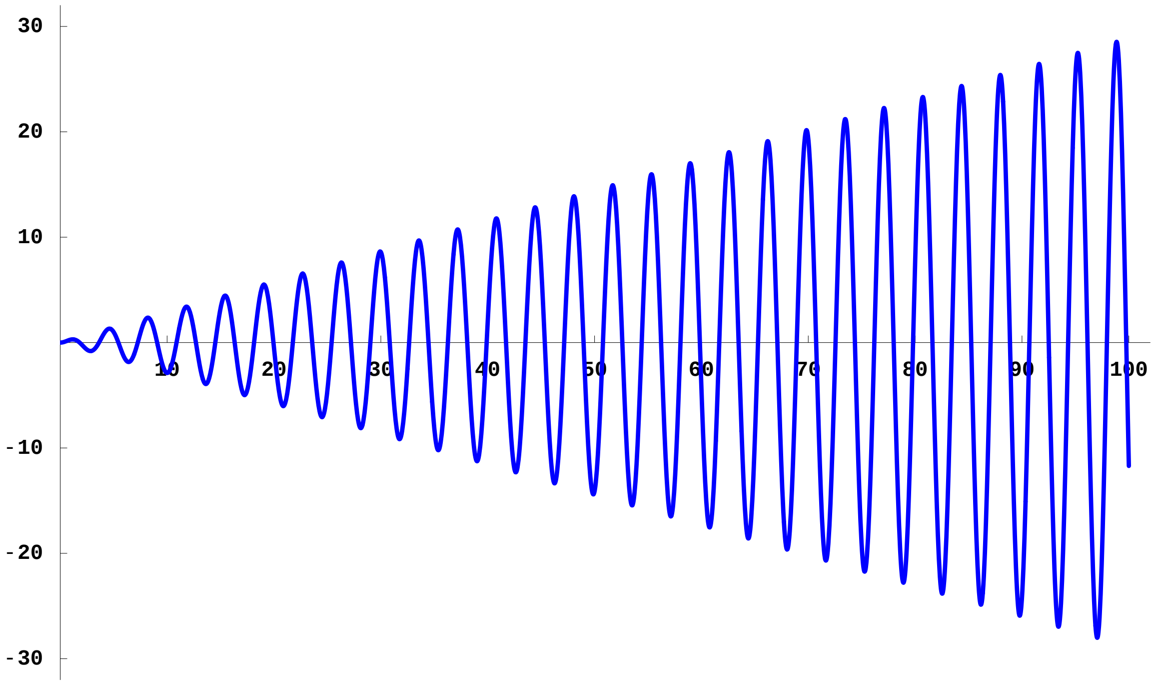
$$\frac{d^2y}{dt^2} + 3y = \cos\sqrt{3}\,t$$

The complexified equation is

$$\frac{d^2y}{dt^2} + 3y = e^{i\sqrt{3}\,t}.$$

What guess should we use?

Here is the graph of $y_p(t)$.



Linearization:

We would like to apply what we know about linear systems to nonlinear systems.

Example. Consider the van der Pol equation

$$\frac{d^2x}{dt^2} + (x^2 - 1)\frac{dx}{dt} + x = 0.$$

The corresponding system is

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= (1 - x^2)y - x.\end{aligned}$$

Let's calculate the equilibria:

Example. Consider the (undamped) pendulum

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0.$$

The corresponding system is

$$\begin{aligned}\frac{d\theta}{dt} &= v\\ \frac{dv}{dt} &= -\sin\theta.\end{aligned}$$

Let's calculate the equilibria:

The linearized system near $(\pi, 0)$ is