MA 226 April 20, 2005

Linearization

Last class we began to apply what we know about linear systems to nonlinear systems.

Linearization Theorem Let $\mathbf{Y}_0$ be an equilibrium point for the nonlinear autonomous system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$$

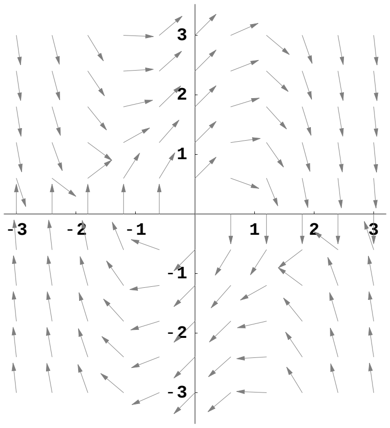
and let

$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}\mathbf{Y}$$

be the corresponding linearized system. If the eigenvalues of $\mathbf{J}$ are not purely imaginary, then the solution curves of the nonlinear system near $\mathbf{Y}_0$ behave in the same qualitative way as the solution curves of the linear system.

Example. Consider the van der Pol equation near the origin. The linearized system is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{Y}.$$

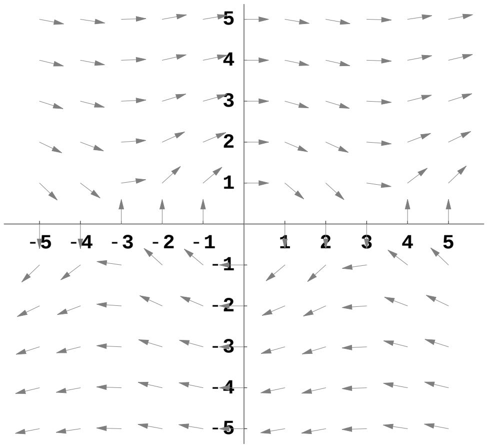


Example. Consider the pendulum equation. The linearized system near $(\pi, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{Y}.$$

The linearized system near $(0, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$

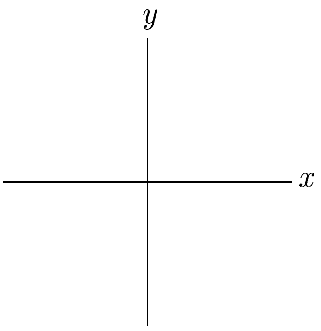


What is special about the case of purely imaginary eigenvalues in the linearization?

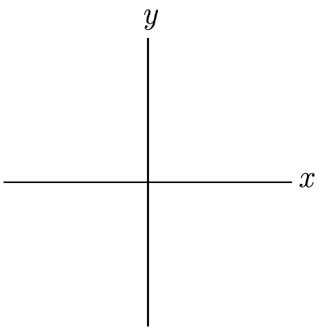
Example. Consider the one-parameter family of systems

$$\begin{aligned} \frac{dx}{dt} &= -y + \alpha x(x^2+y^2) \\ \frac{dy}{dt} &= \quad x + \alpha y(x^2+y^2) \end{aligned}$$

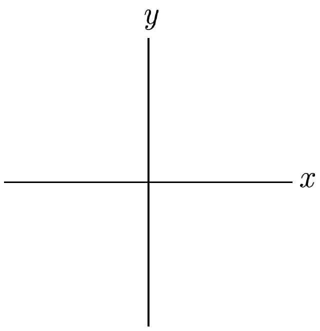
where α is a parameter. Note that $(0,0)$ is always an equilibrium point.



$\alpha < 0$



$\alpha = 0$



$\alpha > 0$