

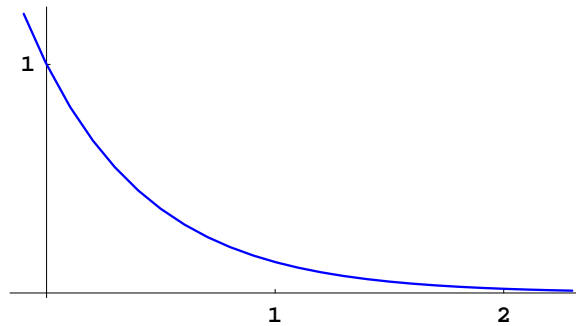
**The Laplace transform**

For the remainder of the semester, we are going to take a somewhat different approach to the solution of differential equations. We are going to study a way of transforming differential equations into algebraic equations.

We begin with a little review of improper integrals.

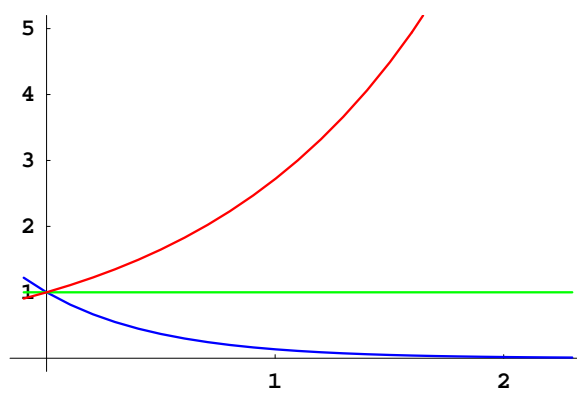
**Example.** Consider the improper integral

$$\int_0^{\infty} e^{-2t} dt.$$



**Example.** Consider the improper integrals

$$\int_0^{\infty} e^{-st} dt.$$



**Definition.** The *Laplace transform* of the function  $y(t)$  is the function

$$Y(s) = \int_0^{\infty} y(t) e^{-st} dt.$$

This transform is an “operator” (a function on functions). It transforms the function  $y(t)$  into the function  $Y(s)$ .

Notation: We often represent this operator using the script letter  $\mathcal{L}$ . In other words,

$$\mathcal{L}[y] = Y.$$

For example,

$$\mathcal{L}[1] = \frac{1}{s}.$$

Note that, even if  $y(t)$  is defined for all  $t$ , the Laplace transform  $Y(s)$  may not be defined for all  $s$ .

**Examples.** Using *Mathematica* to calculate the improper integrals, we see that:

$$\mathcal{L}[\sin t] = \frac{1}{s^2 + 1} \quad \text{for } s > 0,$$

$$\mathcal{L}[e^{2t} \sin 3t] = \frac{3}{s^2 - 4s + 13} \quad \text{for } s > 2$$

$$\mathcal{L}[t^4] = \frac{24}{s^5} \quad \text{for } s > 0$$

$$\mathcal{L}[t \cos \sqrt{2} t] = \frac{s^2 - 2}{(s^2 + 2)^2} \quad \text{for } s > 0$$

$$\mathcal{L}[e^{i\omega t}] = \frac{1}{s - i\omega} \quad \text{for } s > 0$$

**Example.** Let's compute  $\mathcal{L}[e^{at}]$  using the definition and the improper integrals we have already computed:

**Properties of the Laplace transform** There are two properties of the Laplace transform that make it well suited for solving linear differential equations:

1.  $\mathcal{L} \left[ \frac{dy}{dt} \right] = s\mathcal{L}[y] - y(0)$
2.  $\mathcal{L}$  is a linear transform

Both of these properties are extremely important, but the surprising one is #1. Let's consider

$$\mathcal{L} \left[ \frac{dy}{dt} \right] = \int_0^{\infty} \left( \frac{dy}{dt} \right) e^{-st} dt.$$

In fact, before we consider the improper integral, let's apply the method of integration by parts to the indefinite integral

$$\int \left( \frac{dy}{dt} \right) e^{-st} dt.$$

Now let's see how we can use the Laplace transform to solve an initial-value problem.

**Example.** Solve the IVP

$$\frac{dy}{dt} - 3y = e^{2t}, \quad y(0) = 4.$$

1. Transform both sides of the equation:

2. Solve for  $\mathcal{L}[y]$ :

3. Calculate the inverse Laplace transform: