More on discontinuous equations

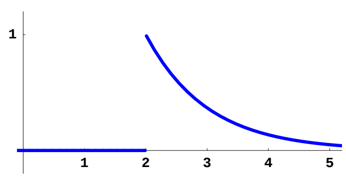
Last class we started discussing differential equations that are "discontinuous" in one way or another. We introduced the Heaviside function $u_a(t)$ (the "light switch" function), and we computed its Laplace transform

$$\mathcal{L}[u_a] = \frac{e^{-as}}{s}.$$

We also started to discuss the rule:

Rule 3: Shifting the t-axis. $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}\mathcal{L}[f]$.

Example. Calculate $\mathcal{L}[g]$ where $g(t) = u_2(t)\, e^{-(t-2)}$.



In this case, $a = 2$ and $f(t) = e^{-t}$. So

$$\mathcal{L}[g] = e^{-2s}\mathcal{L}[f] = e^{-2s}\left(\frac{1}{s+1}\right) = \frac{e^{-2s}}{s+1}.$$

Why does the shifting rule work the way that it does?

Shifting the t-axis. $\mathcal{L}[u_a(t)f(t-a)] =$

Now let's see how we can use these properties of the Laplace transform to solve an initial-value problem that involves discontinuous forcing.

Example. Solve the IVP

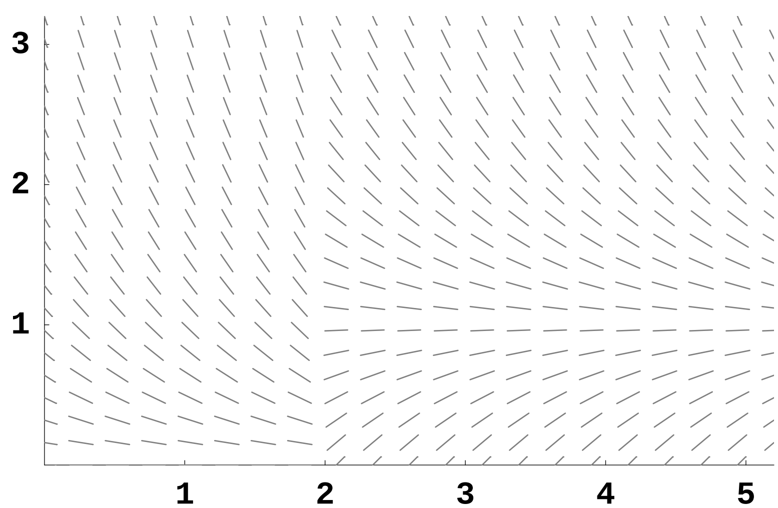
$$\frac{dv}{dt} + v = u_2(t), \quad v(0) = 3.$$

1. Transform both sides of the equation:

2. Solve for $\mathcal{L}[v]$:

3. Calculate the inverse Laplace transform:

Now let's plot this solution using `HPGSolver`.



Laplace transforms and second-order equations

So far we have only applied the Laplace transform to first-order equations. Now we consider second-order equations.

Recall the rule for Laplace transforms of derivatives:

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

What does this say about

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right]?$$

Now that we have this rule, we also need to add to our table of Laplace transforms. Since sine and cosine often appear as parts of the solutions to second-order equations, let's determine their Laplace transforms.

There are a number of ways to compute these transforms—using integration by parts, using Euler's formula, and even using the fact that sine and cosine are solutions to certain very special second-order equations. Last Friday, we saw that

$$\mathcal{L}[e^{i\omega t}] = \frac{1}{s - i\omega}.$$

Let's use this to determine $\mathcal{L}[\sin \omega t]$ and $\mathcal{L}[\cos \omega t]$.

Consider

$$\mathcal{L}[e^{i\omega t}] = \frac{1}{s - i\omega}.$$

Now that we know the transforms of sine and cosine, let's see how we use them.

Example. Compute

$$\mathcal{L}^{-1}\left[\frac{2s+1}{s^2+9}\right].$$

Now for a little practice with the third rule for transforms:

Example. Compute

$$\mathcal{L}^{-1}\left[\frac{8e^{-10s}}{(s^2+9)(s^2+1)}\right].$$