More on Laplace transforms and second-order equations

Last class we started to work with linear second-order differential equations and Laplace transforms. We derived the equation

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2\mathcal{L}[y] - y(0)s - y'(0).$$

We also saw that

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+w^2} \quad \text{and} \quad \mathcal{L}[\sin\omega t] = \frac{w}{s^2+w^2}.$$

Finally, we computed the inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{2s+1}{s^2+9}\right] = 2\cos 3t + \frac{1}{3}\sin 3t.$$

Here's a little more practice with the rule $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}\mathcal{L}[f]$.

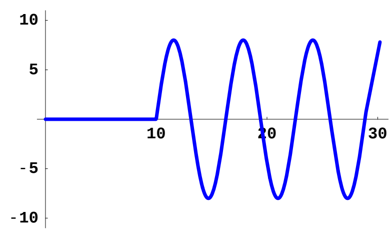
Example. Compute

$$\mathcal{L}^{-1}\left[\frac{8e^{-10s}}{(s^2+9)(s^2+1)}\right].$$

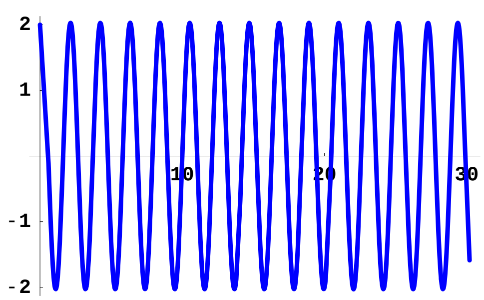
Let's use what we have learned to solve the initial-value problem

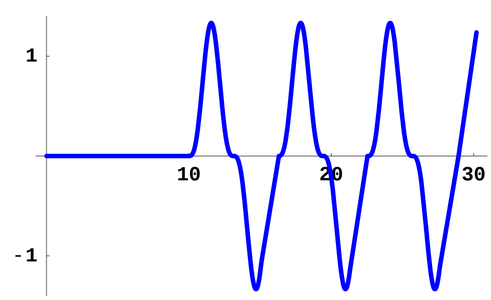
$$\frac{d^2y}{dt^2} + 9y = 8u_{10}(t)\sin(t-10), \quad y(0)=2,\ y'(0)=1.$$

Here is a graph of the forcing function $8u_{10}(t)\sin(t-10)$.

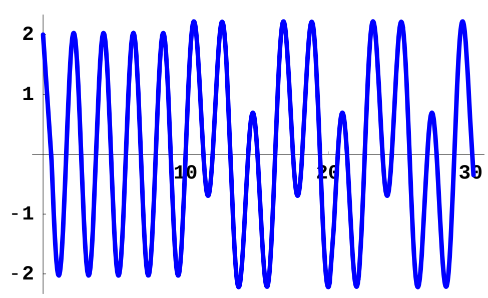


Here are the graphs of the two functions that combine to give us the desired solution.





Here is the graph of the solution



The second-order equations that we have discussed so far are undamped. In order to consider the full range of second-order equations, we need one more property of the transform.

Shifting the s-axis. Let $Y(s)$ denote $\mathcal{L}[y(t)]$. Then

$$\mathcal{L}[e^{at}\, y(t)] =$$

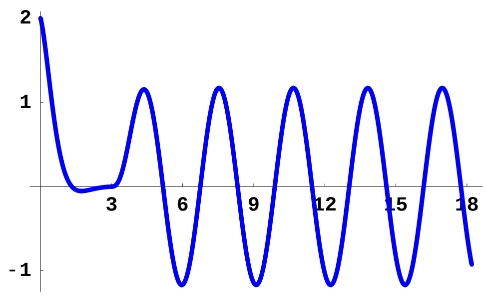
Example. Calculate $\mathcal{L}[e^{-2t} \cos 3t]$.

Example. Calculate $\mathcal{L}^{-1}\left[\dfrac{2s+7}{s^2+4s+7}\right]$.

Let's solve the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 7y = 10\,u_3(t)\,\sin 2(t-3), \quad y(0) = 2,\ y'(0) = -1.$$

Before we get too far into the messy formulas, let's look at the graph of the solution:



Now for the formulas:

1. Transform both sides of the equation:

2. Solve for $\mathcal{L}[y]$:

3. Calculate the inverse Laplace transform:

We calculated

$$\mathcal{L}^{-1}\left[\frac{2s+7}{s^2+4s+7}\right] = 2e^{-2t}\cos\sqrt{3}\,t + \sqrt{3}e^{-2t}\sin\sqrt{3}\,t$$

earlier.

To invert the second term, we take advantage of some algebra done before class:

(a) Partial fractions decomposition:

$$\frac{1}{(s^2+4)(s^2+4s+7)} = \frac{1}{73}\left(\frac{4s+13}{s^2+4s+7} - \frac{4s-3}{s^2+4}\right)$$

(b) Inverse related to the first term:

$$\mathcal{L}^{-1}\left[\frac{4s+13}{s^2+4s+7}\right] = 4e^{-2t}\cos\sqrt{3}\,t + \frac{5\sqrt{3}}{3}e^{-2t}\sin\sqrt{3}\,t$$

(c) Inverse related to the second term:

$$\mathcal{L}^{-1}\left[\frac{4s-3}{s^2+4}\right] = 4\cos 2t - \frac{3}{2}e^{-2t}\sin 2t$$

After we put all of this together, we get the solution

$$y(t) = 2e^{-2t}\cos\sqrt{3}\,t + \sqrt{3}e^{-2t}\sin\sqrt{3}\,t +$$

$$\frac{20}{73}u_3(t)\left(4e^{-2(t-3)}\cos\sqrt{3}(t-3) + \frac{5\sqrt{3}}{3}e^{-2(t-3)}\sin\sqrt{3}(t-3) - 4\cos 2(t-3) + \frac{3}{2}\sin 2(t-3)\right)$$