Application: We can apply what we have learned to the (damped) harmonic oscillator

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

In this case, we are assuming that the parameters m and k are positive and that $b \geq 0$. The characteristic equation $m\lambda^2 + b\lambda + k = 0$ has eigenvalues

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are three cases based on the value of the discriminant $b^2 - 4mk$.

1. $b^2 - 4mk < 0$: In this case, the eigenvalues are complex and can be written as

$$\lambda = \left(-\frac{b}{2m}\right) \pm \left(\frac{\sqrt{4mk - b^2}}{2m}\right) i.$$

   The real part determines the exponential decay rate for solutions, and the imaginary part determines the natural "period" of the solutions.

   There are two subcases:

   (a) $b = 0$: all solutions are periodic. This is the **undamped** case.

   (b) $b \neq 0$: solutions oscillate with a constant frequency, but they decay at an exponential rate. This is the **underdamped** case.

2. $b^2 - 4mk = 0$: The eigenvalue

$$\lambda = -\frac{b}{2m}$$

   is repeated. This is the **critically damped** case. In this case, solutions approach zero as rapidly as possible.

3. $b^2 - 4mk > 0$: The eigenvalues

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

   are both real. Note that

$$0 < b^2 - 4mk < b^2.$$

   Therefore, both eigenvalues are negative, and the equilbrium point at the origin is a (real) sink. The rate of approach to zero by a typical solution is determined by the "slow" eigenvalue. This is the **overdamped** case.

Example. Consider the one-parameter family of equations

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + y = 0.$$

In this case, the characteristic equation is $\lambda^2 + b\lambda + 1 = 0$, and consequently, the eigenvalues are

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4}}{2}.$$

The value $b = 2$ is the critical value for this family.

We can see the progression from underdamped to critically damped to overdamped with a Quicktime animation that I have posted on the web site.

The trace-determinant plane

There is a nice geometric object called the trace-determinant plane that organizes the various types of 2×2 linear systems.

Consider the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let's calculate the characteristic polynomial of $\mathbf{A}$:

Conclusion: The eigenvalues of any 2×2 matrix are determined by the trace and the determinant of $\mathbf{A}$. We have

$$\lambda = \frac{(\mathrm{tr}\mathbf{A}) \pm \sqrt{(\mathrm{tr}\mathbf{A})^2 - 4(\det \mathbf{A})}}{2}.$$

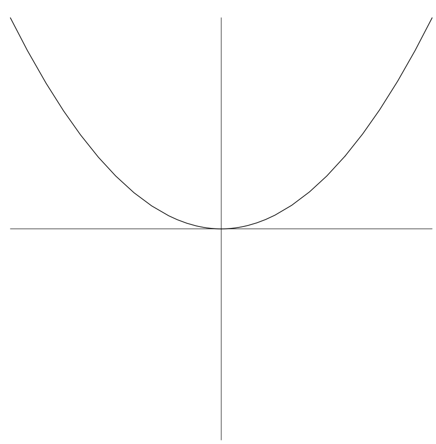
Summary of Phase Portraits

Assume $\det \mathbf{A} \neq 0$. Then zero is not an eigenvalue of $\mathbf{A}$.

1. Real and distinct eigenvalues
   (a) sink
   (b) saddle
   (c) source
2. Complex eigenvalues
   (a) spiral sink
   (b) center
   (c) spiral source
3. Real and repeated eigenvalues
   (a) sink with one eigenline in the phase portrait
   (b) source with one eigenline in the phase portrait
   (c) sink where every solution is a straight-line solution
   (d) source where every solution is a straight-line solution

What if $\det \mathbf{A} = 0$?

We can organize these different types using a plane with unusual coordinate axes.



You can turn on the trace-determinant plane in the `LinearPhasePortraits` tool.

Example. Consider the one-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 1 \\ 0 & d \end{pmatrix} \mathbf{Y}.$$