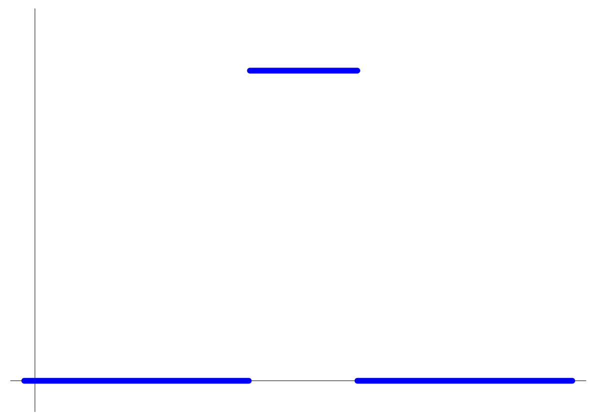
Dirac Delta Function. The Dirac delta "function" $\delta_a(t)$ is used to model impulse forcing.

Suppose we want to model a unit force that is applied instantaneously at time $t = a$. We begin with the function

$$g_{\Delta t}(t) = \begin{cases} \dfrac{1}{\Delta t}, & \text{if } a - \dfrac{\Delta t}{2} \le t < a + \dfrac{\Delta t}{2}; \\ 0, & \text{otherwise.} \end{cases}$$



We can write $g_{\Delta t}$ in terms of the Heaviside function. We get

$$g_{\Delta t} = \frac{1}{\Delta t}\left(u_{a-\frac{\Delta t}{2}} - u_{a+\frac{\Delta t}{2}}\right).$$

Now let's calculate the Laplace transform of $g_{\Delta t}$:

Finally we take the limit as $\Delta t \to 0$.

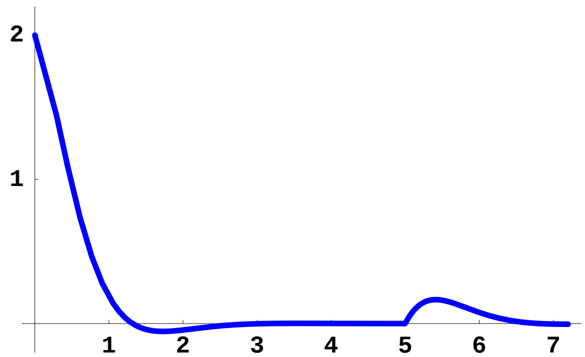
Dirac Delta Function. The Dirac delta function $\delta_a(t)$ is the "function" such that

$$\mathcal{L}[\delta_a] = e^{-as}.$$

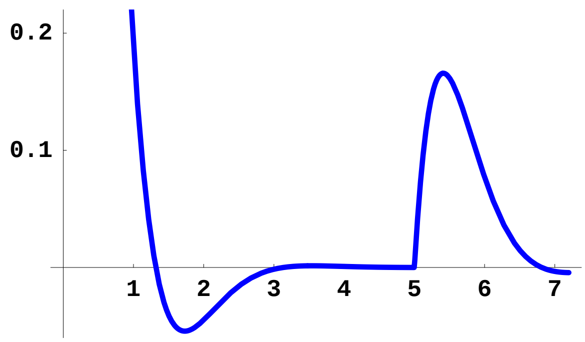
Example. Consider the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 7y = \delta_5(t), \quad y(0) = 2,\ y'(0) = -1.$$

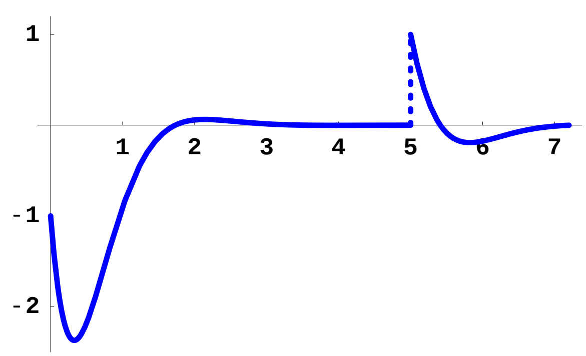
Here is the graph of the solution:



We enlarge the scale on the vertical axis:



Here is the graph of its derivative:



Summary of transform rules and table of standard transforms

Here are important selections from the summary on page 602 in your text.

$y(t)$	$Y(s) = \mathcal{L}[y]$	
$y(t) = 1$	$Y(s) = \dfrac{1}{s}$	$(s > 0)$
$y(t) = e^{at}$	$Y(s) = \dfrac{1}{s-a}$	$(s > a)$
$y(t) = u_a(t)$	$Y(s) = \dfrac{e^{-as}}{s}$	$(s > 0)$
$y(t) = \cos \omega t$	$Y(s) = \dfrac{s}{s^2 + \omega^2}$	$(s > 0)$
$y(t) = \sin \omega t$	$Y(s) = \dfrac{\omega}{s^2 + \omega^2}$	$(s > 0)$
$y(t) = \delta_a(t)$	$Y(s) = e^{-as}$	

Properties of the Laplace Transform

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

$$\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2]$$

$$\mathcal{L}[\alpha\, y] = \alpha \mathcal{L}[y] \text{ for any constant } \alpha$$

$$\mathcal{L}[u_a(t) y(t-a)] = e^{-as}\mathcal{L}[y]$$

$$\mathcal{L}[e^{at} y(t)] = Y(s-a) \text{ where } Y = \mathcal{L}[y]$$

Some people like to memorize a few more entries such as

$$\mathcal{L}[e^{at}\cos\omega t] = \frac{s-a}{(s-a)^2+\omega^2},$$

but I prefer to use the last rule (shifting the s-axis). Also, the rule for $\mathcal{L}[dy/dt]$ in terms of $\mathcal{L}[y]$ yields

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2\mathcal{L}[y] - y(0)s - y'(0).$$

Warning: Just because you can solve a linear differential equation with the Laplace transform does not mean that you should forget what you learned in previous parts of the course. The transform method is particularly well suited for differential equations with discontinuous and impulse forcing.