More on linearity principles

Recall the Linearity Principle for homogeneous equations.

Linearity Principle. If $y_h(t)$ is a solution of a homogeneous linear differential equation

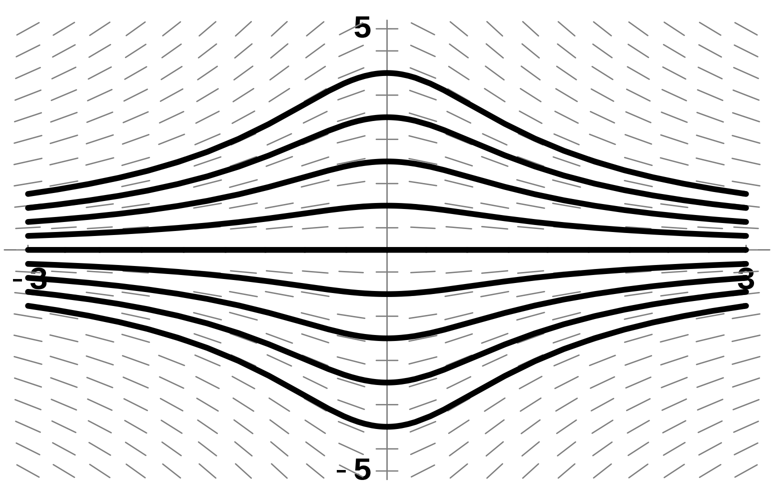
$$\frac{dy}{dt} = a(t)y,$$

then any *constant* multiple $y_k(t) = ky_h(t)$ of $y_h(t)$ is also a solution. In other words, given a constant $k \neq 1$ and a solution $y_h(t)$, we obtain another solution by multiplying $y_h(t)$ by k.

Example. $\dfrac{dy}{dt} = \dfrac{-ty}{1+t^2}$

Last class we calculated the general solution

$$y(t) = \frac{y_0}{\sqrt{1+t^2}}.$$



Note that the Linearity Principle is not true for nonlinear equations. For example, consider

$$\frac{dy}{dt} = y^2.$$

Check that one solution is

$$y_1(t) = \frac{1}{1-t},$$

and then check that

$$y_2(t) = 2y_1(t) = \frac{2}{1-t}$$

is not a solution.

There is a similar "linearity" principle for nonhomogeneous linear equations:

Extended Linearity Principle For First-Order Equations. Consider a first-order, nonhomogeneous, linear equation

$$\frac{dy}{dt} = a(t)y + b(t)$$

and its associated homogeneous equation

$$\frac{dy}{dt} = a(t)y.$$

1. If $y_h(t)$ is any solution of the homogeneous equation and $y_p(t)$ ("p" for particular solution) is *any* solution of the nonhomogeneous equation, then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

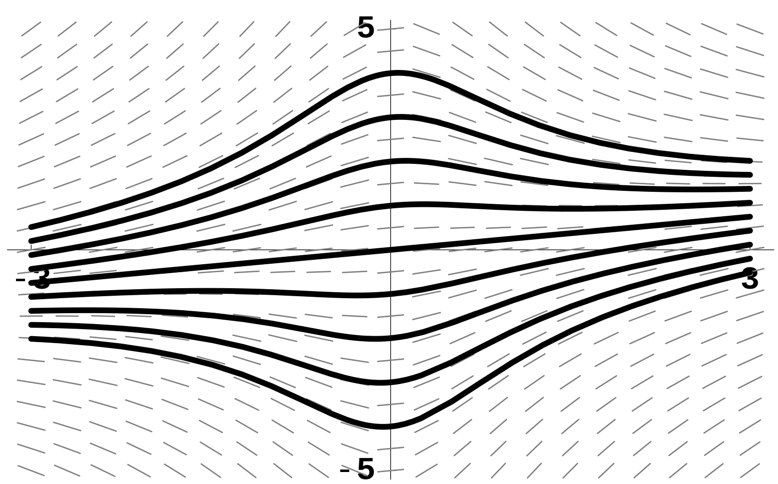
2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

Therefore, if $y_h(t)$ is nonzero, $ky_h(t) + y_p(t)$ is the general solution of the nonhomogeneous equation.

We can paraphrase the Extended Linearity Principle by saying that:

> The general solution of a nonhomogeneous linear equation consists of the sum of *any* particular solution of the nonhomogeneous equation and the general solution of the associated homogeneous equation.

Example. $\dfrac{dy}{dt} = \dfrac{-ty}{1+t^2} + \dfrac{2t^2+1}{4t^2+4}$



The Extended Linearity Principle provides one way to solve nonhomogeneous equations:

Method of the Lucky Guess:

1. Solve the associated homogeneous equation.
2. Guess one solution to the nonhomogeneous equation.

How do we go about guessing a solution? This question is best answered by doing a number of typical examples.

Example 1. $\dfrac{dy}{dt} = -2y + 3e^{-t/2}$

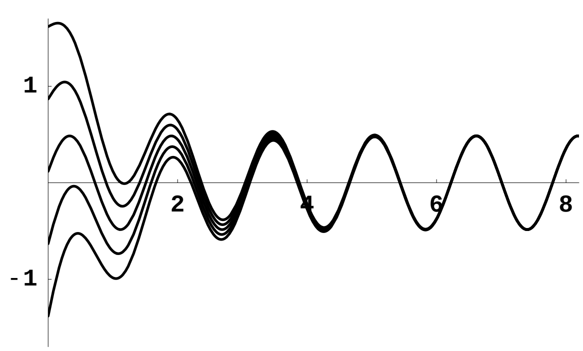
1. General solution of the associated homogeneous equation:

2. Particular solution of the nonhomogeneous equation:

Example 2. $\dfrac{dy}{dt} = -y + 2\cos 4t$

1. General solution of the associated homogeneous equation:

2. Particular solution of the nonhomogeneous equation:



Example 3. $\dfrac{dy}{dt} = -3y + 2e^{-3t}$

1. General solution of the associated homogeneous equation:

2. Particular solution of the nonhomogeneous equation (trick question):