Two examples from last class

Example. The predator-prey system, 2D first-order autonomous system,

$$\frac{dR}{dt} = aR - bRF$$
$$\frac{dF}{dt} = -cF + dRF.$$

Example. The mass-spring system, a second-order autonomous equation,

$$m\frac{d^2y}{dt^2} + ky = 0.$$

Although they seem quite different, they have more in common than one might think. In particular, right at the end of class, I showed how the mass-spring system can also be written as a first-order system by introducting the "new" variable v (which is just dy/dt). We get

In what ways is the mass-spring system similar to the predator-prey system?

An initial condition for the predator-prey system is a pair (R_0, F_0) of population values.

An initial condition for the mass-spring system is also a pair (y_0, v_0). The first number indicates the initial displacement and the second number indicates the initial velocity.

Since the mass-spring system can be expressed as a system, all of the terms that we discussed last class for the predator-prey system apply to the mass-spring system as well (equilibrium solutions, component graphs, phase plane, solution curve, phase portrait, . . .). There are two animations on the class web site that illustrate these ideas for the mass-spring system.

One way that the two systems differ is by the fact that we can find formulas for the solutions of the mass-spring system but not for the predator-prey system.

For example, consider the special case of the mass-spring system where $k = m$. We get

$$\frac{d^2y}{dt^2} = -y,$$

and we can guess some solutions to this equation:

The equivalent system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -y$$

has one $(y(t), v(t))$ pair of solutions for each solution to the second-order equation:

What are the initial conditions for these solutions?

Where do we go from here?

1. Vector fields (similar to slope fields)
2. Two analytic techniques (more guessing)
3. Euler's method again
4. Some theory (Existence and Uniqueness)
5. Linear systems and equations (Chapter 3)

The vector field of an autonomous system

We get a better geometric understanding of the solutions of a first-order system if we convert the system into a vector equation.

Example 1. Once again we consider a simple mass-spring system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -y.$$

By some intelligent guessing we know a few solutions. One is $(y(t), v(t)) = (\cos t, -\sin t)$. We rewrite this system and the solution in terms of vectors:

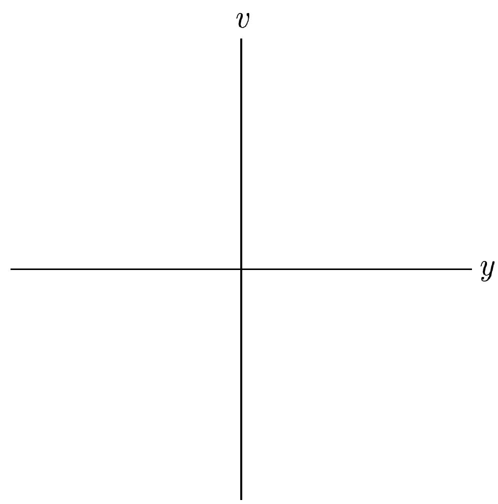
Now for the geometric interpretation of

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}),$$

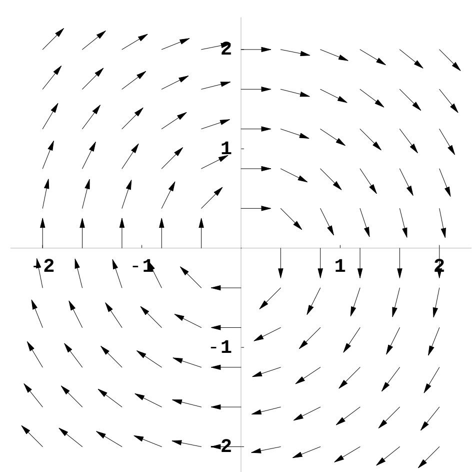
where

$$\mathbf{F}(\mathbf{Y}) = \mathbf{F}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -y \end{pmatrix}.$$

We use `HPGSystemSolver` to help visualize the vector field and the solutions.



The direction field associated with this system is



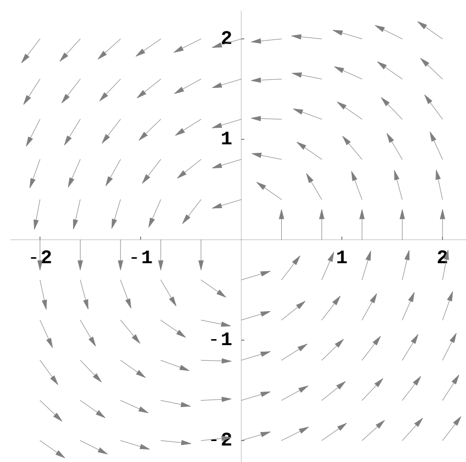
Example 2. Consider the system

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x - 0.3y.$$

The vector field associated with this system is

$$\mathbf{F}(\mathbf{Y}) = \mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x - 0.3y \end{pmatrix}.$$

Here's the direction field:



Consider the following 8 first-order systems:

1. $\dfrac{dx}{dt} = -x$, $\dfrac{dy}{dt} = y^2 - 1$

2. $\dfrac{dx}{dt} = -2x$, $\dfrac{dy}{dt} = -y$

3. $\dfrac{dx}{dt} = -x - 2y$, $\dfrac{dy}{dt} = y$

4. $\dfrac{dx}{dt} = 1 - y$, $\dfrac{dy}{dt} = 1 + x$

5. $\dfrac{dx}{dt} = x$, $\dfrac{dy}{dt} = 2x - y$

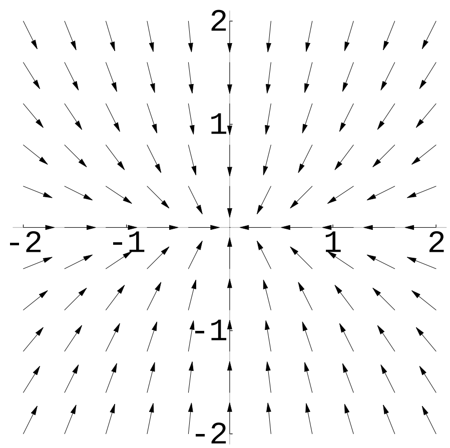
6. $\dfrac{dx}{dt} = y - 1$, $\dfrac{dy}{dt} = -1 - x$

7. $\dfrac{dx}{dt} = -x$, $\dfrac{dy}{dt} = -2y$

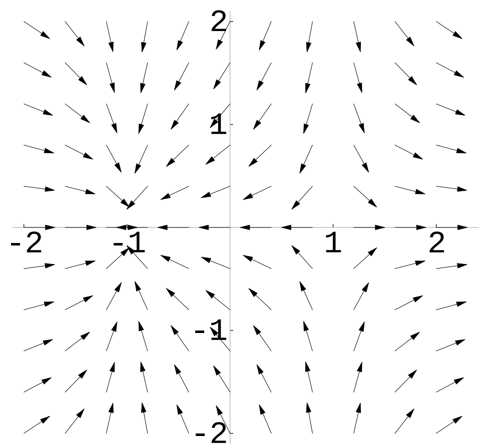
8. $\dfrac{dx}{dt} = x^2 - 1$, $\dfrac{dy}{dt} = -y$

Four of the associated direction fields are shown below. Pair the direction fields with their associated systems. Provide a brief justification for your choice.

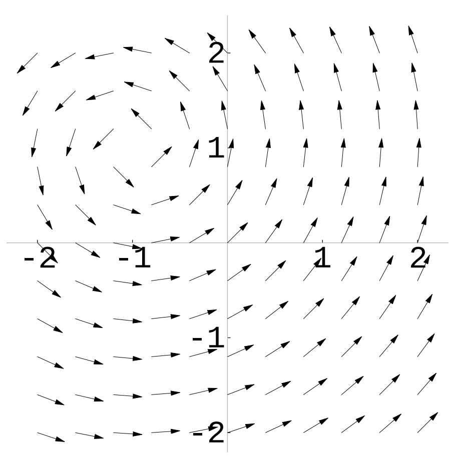
Direction Field A



Direction Field B



Direction Field C



Direction Field D