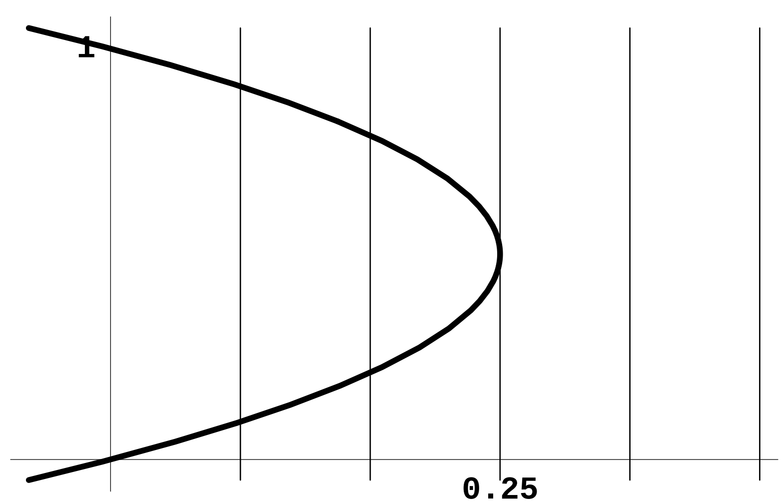
More on bifurcations

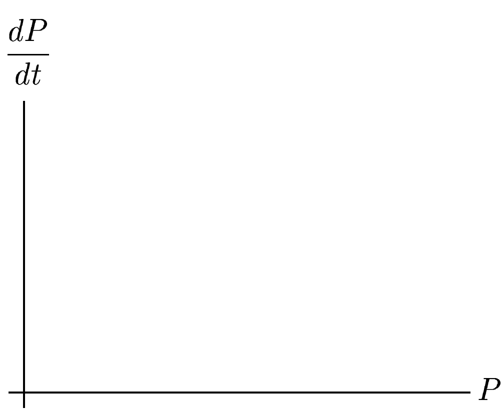
Example. $\dfrac{dy}{dt} = y(1-y) - a$

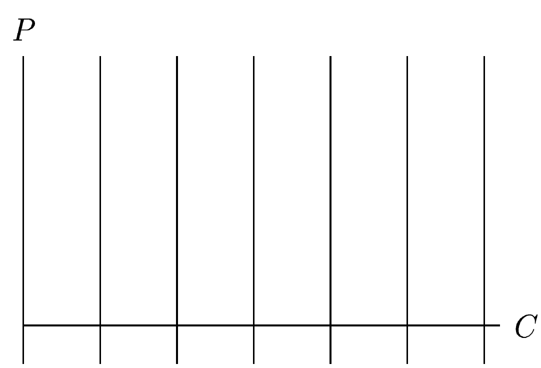
There is a tool in `DETools` called `PhaseLines`, and it helps us analyze phase lines and various graphs as we vary certain parameters (the parameter a in this case).



Now let's sketch and interpret the bifurcation diagram for the logistic population model with constant harvesting

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - C.$$





Linear Differential Equations

A first-order differential equation

$$\frac{dy}{dt} = f(t, y)$$

with independent variable t and dependent variable y is **linear** if it can be written as

$$\frac{dy}{dt} = a(t)y + b(t).$$

In other words, the *dependent* variable only appears linearly in the equation.

Linear differential equations:

$$\frac{dy}{dt} = 5y$$

$$\frac{dy}{dt} = (\cos t)y$$

$$\frac{dy}{dt} = y - t^2$$

Nonlinear differential equations:

$$\frac{dy}{dt} = t \cos y$$

$$\frac{dy}{dt} = y^2 - t$$

The linear differential equation

$$\frac{dy}{dt} = a(t)y + b(t)$$

is **homogeneous** if $b(t) = 0$ for all t. Otherwise, it is **nonhomogeneous**. (Some people use the term inhomogeneous.)

Where have we seen homogeneous linear differential equations before?

Linearity Principles

Why are linear equations so much more amenable to analytic techniques than nonlinear equations? The reason is that their solutions satisfy important linearity principles.

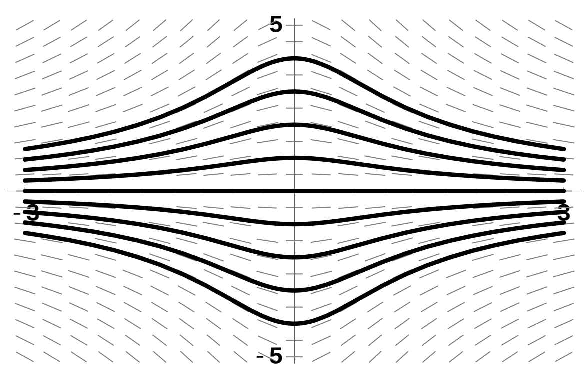
Let's begin with homogeneous linear equations:

Linearity Principle. If $y_h(t)$ is a solution of a homogeneous linear differential equation

$$\frac{dy}{dt} = a(t)y,$$

then any *constant* multiple $y_k(t) = ky_h(t)$ of $y_h(t)$ is also a solution. In other words, given a constant $k \neq 1$ and a solution $y_h(t)$, we obtain another solution by multiplying $y_h(t)$ by k.

Example. $\frac{dy}{dt} = \frac{-ty}{1+t^2}$



Note that the Linearity Principle is not true for nonlinear equations. For example, consider

$$\frac{dy}{dt} = y^2.$$

Check that one solution is

$$y_1(t) = \frac{1}{1-t},$$

and then check that

$$y_2(t) = 2y_1(t) = \frac{2}{1-t}$$

is not a solution.

There is a similar "linearity" principle for nonhomogeneous linear equations:

Extended Linearity Principle For First-Order Equations. Consider a first-order, nonhomogeneous, linear equation

$$\frac{dy}{dt} = a(t)y + b(t)$$

and its associated homogeneous equation

$$\frac{dy}{dt} = a(t)y.$$

1. If $y_h(t)$ is any solution of the homogeneous equation and $y_p(t)$ ("p" for particular solution) is *any* solution of the nonhomogeneous equation, then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

Therefore, if $y_h(t)$ is nonzero, $ky_h(t) + y_p(t)$ is the general solution of the nonhomogeneous equation.