Linear systems—a brief review

A linear system (with constant coefficients) can be written as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

where $\mathbf{A}$ is a square matrix of constants (the coefficients). For us, $\mathbf{A}$ will be a 2×2 matrix.

Using the Linearity Principle, we can produce many solutions from just a few:

If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, then

$$k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$$

is a solution for any choice of constants k_1 and k_2.

Recall the example we did before break:

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}\mathbf{Y}$$

and the two solutions

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}.$$

Any linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ is also a solution to the system. Moreover, since these two solutions are linearly independent, the general solution of this system is

$$\begin{aligned} \mathbf{Y}(t) &= k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t) \\ &= k_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

The special solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ come from the eigenvalues and eigenvectors of the matrix. That is, if a nonzero vector $\mathbf{Y}_0$ satisfies

$$\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$$

for some scalar λ, then we get a solution whose solution curve is a half line in the phase plane.

"Straight-line" Solutions. Suppose that

$$\mathbf{A}\mathbf{Y}_0 = \lambda \mathbf{Y}_0$$

for some nonzero vector $\mathbf{Y}_0$ and some scalar λ. Then the function

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$$

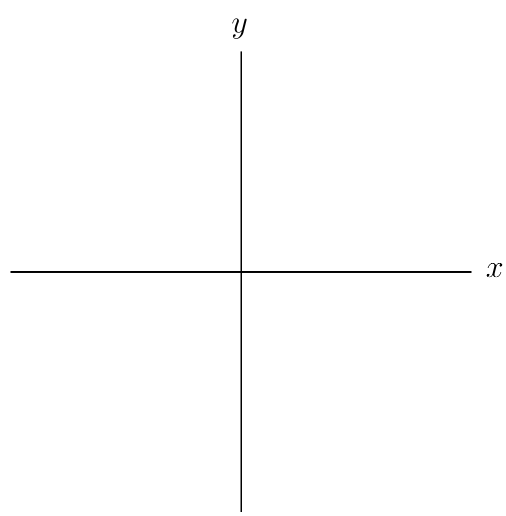
is a solution to the linear differential equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}.$$

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$

First let's see what `MatrixFields` tells us about the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.



Aside from the theory of algebraic linear equations

For what matrices $\mathbf{B}$ does the equation $\mathbf{BY} = \mathbf{0}$ have nontrivial solutions?

Singular Matrices. The matrix equation

$$\mathbf{BY} = \mathbf{0}$$

has nontrivial solutions $\mathbf{Y}$ if and only if $\det \mathbf{B} = 0$.

Note: Most matrices are nonsingular.

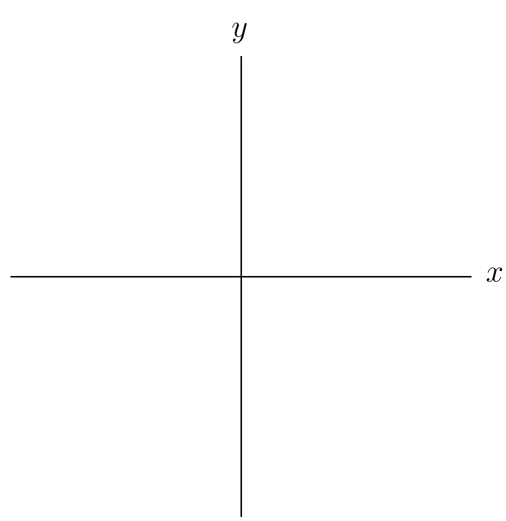
Now let's use this theorem to find eigenvalues and eigenvectors:

Back to the example:

Example. Find the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$

Using `HPGSystemSolver`, we plot the phase portrait for this system.



Facts about eigenvalues and eigenvectors: Given a 2×2 matrix $\mathbf{A}$,

1. The characteristic equation can have two real roots, one real root of multiplicity two, or two complex conjugate roots.
2. Given an eigenvector $\mathbf{Y}_0$ associated to an eigenvalue λ, then any nonzero scalar multiple $\mathbf{Y}_0$ is also an eigenvector associated to λ.
3. Eigenvectors associated to distinct eigenvalues are linearly independent.