A little more on sketching component graphs

Once again, consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$

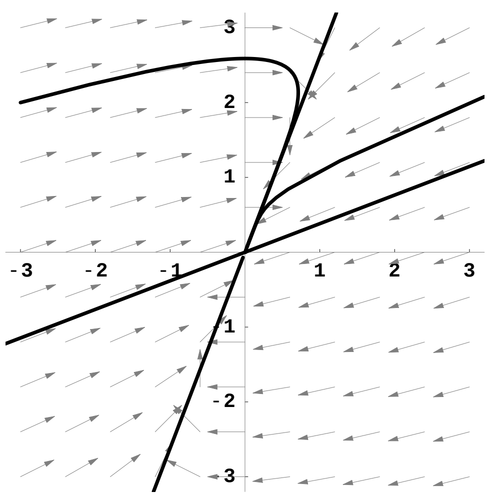
For this example, the eigenvalues are

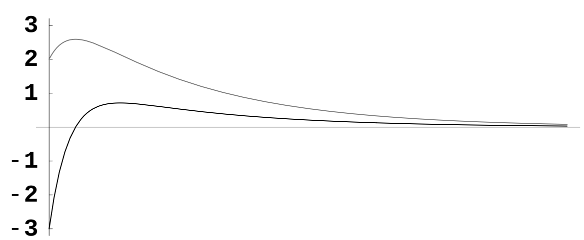
$$\lambda = \frac{-3 \pm \sqrt{5}}{2}.$$

Both are negative.

The eigenline corresponding to the slow eigenvalue $\lambda_2 = \frac{1}{2}(-3 + \sqrt{5})$ has slope that is approximately 2.6, and the eigenline corresponding to the fast eigenvalue $\lambda_1 = \frac{1}{2}(-3 - \sqrt{5})$ has slope that is approximately 0.4.

Last class we sketched the $x(t)$- and $y(t)$-graphs that correspond to the initial condition $(-3, 2)$. Now let's sketch the $x(t)$- and $y(t)$-graphs that correspond to the initial condition $(3, 2)$.





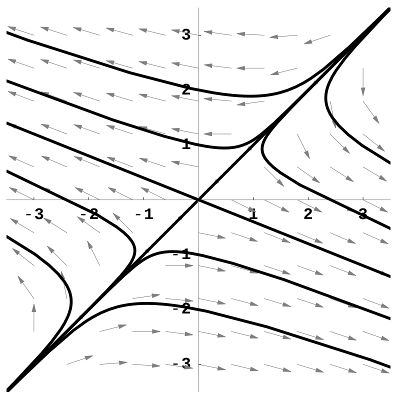
More on phase portraits for distinct real eigenvalues

Case 2: $\lambda_1 < 0 < \lambda_2$.

We did this case last class

Example. Consider

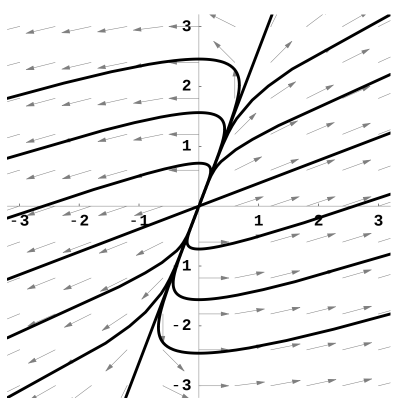
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$



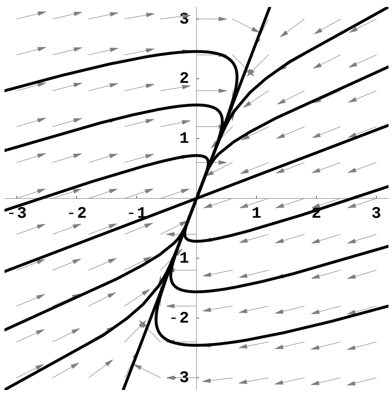
Case 3: $0 < \lambda_1 < \lambda_2$.

Example. Consider

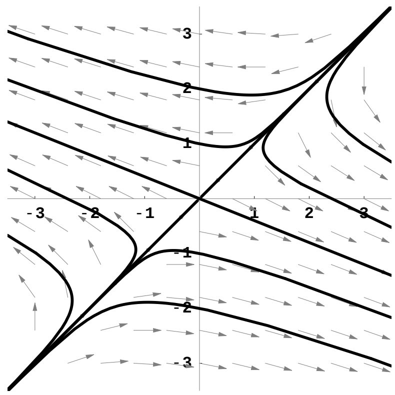
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{Y}.$$



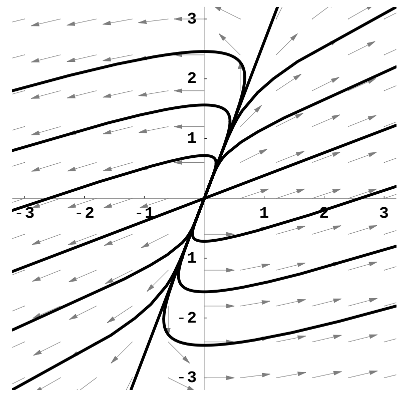
Summary for real and distinct eigenvalues



sink ($\lambda_1 < \lambda_2 < 0$)



saddle ($\lambda_1 < 0 < \lambda_2$)



source ($0 < \lambda_1 < \lambda_2$)

Complex eigenvalues

What happens if the eigenvalues of the system are complex numbers?

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{Y}.$$

Let's see that happens if we take a look at this system using `MatrixFields` and then we'll compute the eigenstuff for this matrix.

We now have a complex-valued solution of the form

$$\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}.$$

We are interested in real-valued solutions. What good is this complex-valued solution?

Once again Euler comes to the rescue: Remember the power series for the exponential function? It is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots .$$

Let's use this series where $x = bi$.

We will use Euler's formula applied to the complex-valued function

$$e^{(a+bi)t}.$$

But why does this help us solve our differential equation?

Theorem. Consider

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

where $\mathbf{A}$ is a matrix with real entries. If $\mathbf{Y}_c(t)$ is a complex-valued solution, then both

$$\mathrm{Re}\mathbf{Y}_c(t) \quad \text{and} \quad \mathrm{Im}\mathbf{Y}_c(t)$$

are real-valued solutions, and they are linearly independent.