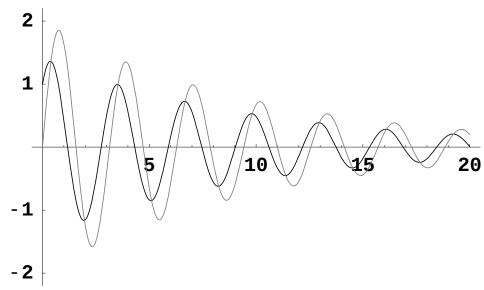
Frequency

Before we tackle repeated eigenvalues, I want to make a few comments about frequency. On Monday, we discussed solutions that decay but oscillate as they do so. Here's the graph of one:



Frequency versus period: This solution is not periodic in the strict sense. There is no time T such that

$$x(t+T) = x(t) \quad \text{and} \quad y(t+T) = y(t)$$

for all t. However, there is a period associated to these solutions. In the text, we call this the **natural period** of the solutions.

Perhaps it is best to think about these solutions as oscillating solutions that are decaying over time and to measure the oscillations in terms of their **frequency**.

Definition. The *frequency* F of an oscillating function $g(t)$ is the number of cycles that $g(t)$ makes in one unit of time.

Suppose that $g(t)$ is oscillating periodically with "period" T. What is its frequency F?

Example. Consider the standard sinusoidal functions $g(t) = \cos \beta t$ and $g(t) = \sin \beta t$.

Suppose we measure frequency in radians rather than in cycles. This measure of frequency is often called **angular frequency**. Let's denote the angular frequency by f. Then

$$f = 2\pi F.$$

Repeated eigenvalues: Sometimes the characteristic polynomial has the same real root twice. When this happens, we say that the eigenvalues are "repeated."

Example. Consider the linear system $d\mathbf{Y}/dt = \mathbf{AY}$ where

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{A}$ is $(\lambda - 3)^2$, so there is only one eigenvalue, $\lambda = 3$. Let's calculate the associated eigenvectors and straight-line solutions:

But we already know how to solve this system. How?

We obtain the general solution

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{3t} + 2y_0 t e^{3t} \\ y_0 e^{3t} \end{pmatrix}$$
$$= e^{3t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^{3t} \begin{pmatrix} 2y_0 \\ 0 \end{pmatrix}.$$

Note that this general solution is *not* written as a linear combination. Every nontrivial solution contains the first term, and most solutions contain both terms.

We use this result to motivate a different technique that we use to solve systems with repeated eigenvalues. We use a guessing technique where we guess a solution of the form

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}_0 + t e^{\lambda t}\mathbf{V}_1.$$

Note that the initial condition for this solution is $\mathbf{V}_0$.

Fact from linear algebra: If $\mathbf{A}$ is a 2×2 matrix with a repeated eigenvalue λ and $\mathbf{V}_0$ is any vector, then either

1. $(\mathbf{A} - \lambda\mathbf{I})\mathbf{V}_0 = \mathbf{0}$ (in other words, $\mathbf{V}_0$ is an eigenvector), or
2. the vector $\mathbf{V}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{V}_0$ is an eigenvector of $\mathbf{A}$.

Example. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 + 4\lambda + 4$, so $\lambda = -2$ is a repeated eigenvalue.

What is the long-term behavior of a system with a repeated, negative eigenvalue?

It is interesting to look at this example using two of the tools on the CD. Using `LinearPhasePortraits`, we can see that this system is on the boundary between spiral sinks and real sinks.

We can also use `HPGSystemSolver` to plot the phase portrait.