Second-order, linear equations: Last class we saw how the guessing technique ($y = e^{\lambda t}$) for

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

with its characteristic equation $a\lambda^2 + b\lambda + c = 0$ produced the eigenvalues for the corresponding system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{c}{a}y - \frac{b}{a}v. \end{aligned}$$

Useful observation: If λ is an eigenvalue, the vector

$$\mathbf{Y}_0 = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

is *always* an associated eigenvector.

Let's see what that tells us about solutions to the second-order equation. There are three cases:

1. Two real, distinct, nonzero eigenvalues λ_1 and λ_2:

2. A complex-conjugate pair of eigenvalues $\lambda = \alpha \pm i\beta$, with $\beta \neq 0$:

3. One nonzero real eigenvalue λ of multiplicity two:

Conclusion: We can determine the general solution of a second-order homogeneous linear equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

immediately from the characteristic equation $a\lambda^2 + b\lambda + c = 0$.

YOU DO NOT NEED TO CALCULATE THE EIGENVECTORS OR EVEN REDUCE TO A FIRST-ORDER SYSTEM if you simply want to produce the general solution of a second-order linear equation.

Application: We can apply what we have learned to the (damped) harmonic oscillator

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

In this case, we are assuming that the parameters m and k are positive and that $b \geq 0$. The characteristic equation $m\lambda^2 + b\lambda + k = 0$ has eigenvalues

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are three cases based on the value of the discriminant $b^2 - 4mk$.

1. $b^2 - 4mk < 0$

2. $b^2 - 4mk = 0$

3. $b^2 - 4mk > 0$

We can see the progression from underdamped to critically damped to overdamped with a Quicktime animation I have posted on the web site.