The Linearity Principle

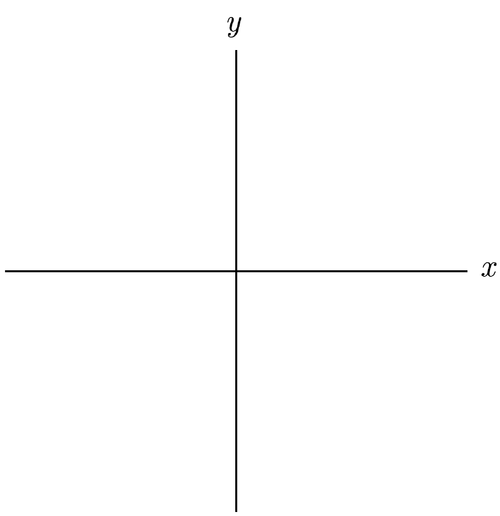
**Example.** Let's return to the example

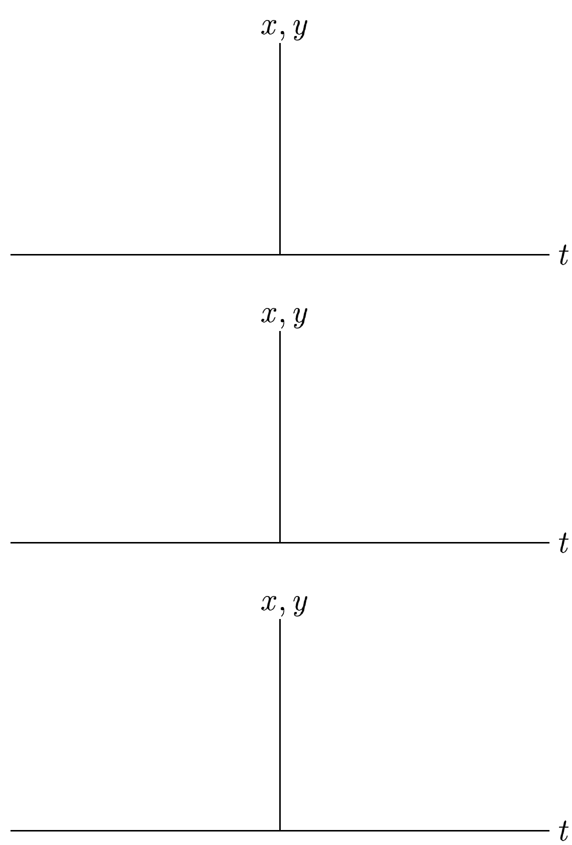
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}.$$

Last class we discussed the three solutions

$$\mathbf{Y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_2(t) = e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{Y}_3(t) = \begin{pmatrix} e^t + e^{-t} \\ e^t \end{pmatrix}.$$

Note that $\mathbf{Y}_3(t) = \mathbf{Y}_1(t) + \mathbf{Y}_2(t)$. Let's see what happens when we graph these solutions.



We can use the Linearity Principle to solve initial-value problems.

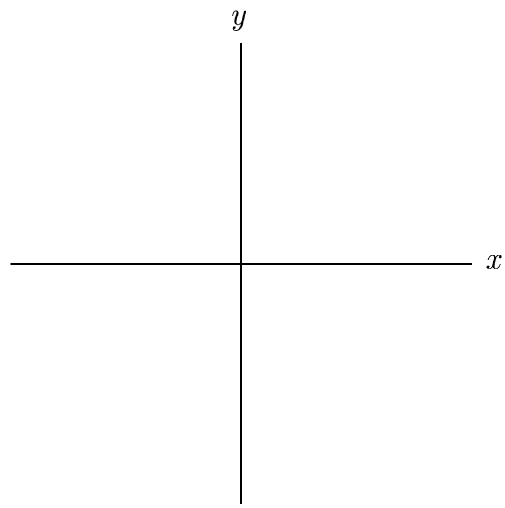
**Example.** Solve

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

In general, how many solutions do we need to be able to solve any initial-value problem?

Question: How do we find two linearly independent solutions?

What is special about the two solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ in the previous example?



We want nonzero initial conditions $\mathbf{Y}_0$ (vectors) so that

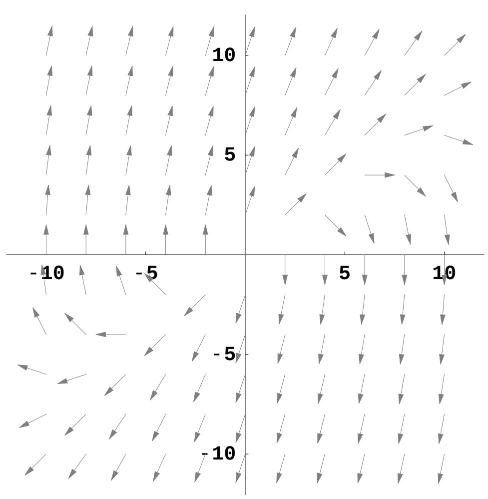
$$\mathbf{A}\mathbf{Y}_0 = \lambda \mathbf{Y}_0$$

for some scalar $\lambda$.

**Terminology:** The scalar $\lambda$ is called an *eigenvalue* of the matrix $\mathbf{A}$ and the vector $\mathbf{Y}_0$ is called an *eigenvector* associated to the eigenvalue $\lambda$.

**Example.** Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{Y}.$$



**"Straight-line" Solutions.** Suppose that

$$\mathbf{A}\mathbf{Y}_0 = \lambda \mathbf{Y}_0$$

for some nonzero vector $\mathbf{Y}_0$ and some scalar $\lambda$. Then the function

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$$

is a solution to the linear differential equation $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$.