The trace-determinant plane

There is a nice geometric object called the trace-determinant plane that organizes the various types of 2×2 linear systems.

Consider the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let's calculate the characteristic polynomial of $\mathbf{A}$:

Conclusion: The eigenvalues of any 2×2 matrix are determined by the trace and the determinant of $\mathbf{A}$. We have

$$\lambda = \frac{(\mathrm{tr}\mathbf{A}) \pm \sqrt{(\mathrm{tr}\mathbf{A})^2 - 4(\det \mathbf{A})}}{2}.$$

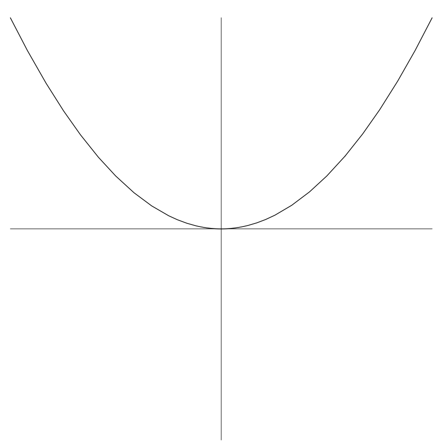
Summary of Phase Portraits

Assume $\det \mathbf{A} \neq 0$. Then zero is not an eigenvalue of $\mathbf{A}$.

1. Real and distinct eigenvalues
   (a) sink
   (b) saddle
   (c) source
2. Complex eigenvalues
   (a) spiral sink
   (b) center
   (c) spiral source
3. Real and repeated eigenvalues
   (a) sink with one eigenline in the phase portrait
   (b) source with one eigenline in the phase portrait
   (c) sink where every solution is a straight-line solution
   (d) source where every solution is a straight-line solution

What if $\det \mathbf{A} = 0$?

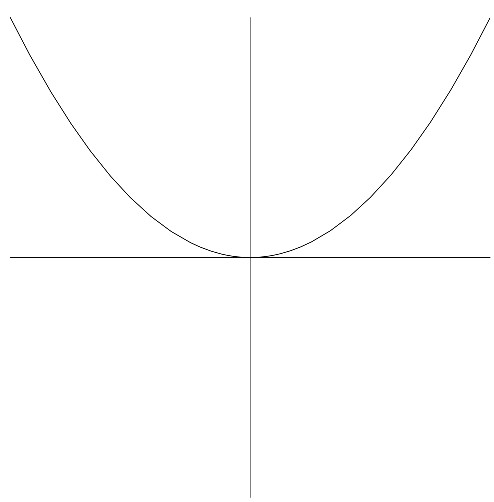
We can organize these different types using a plane with unusual coordinate axes.



You can turn on the trace-determinant plane in the `LinearPhasePortraits` tool.

Example. Consider the one-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 1 \\ 0 & d \end{pmatrix} \mathbf{Y}.$$



Forced equations

For the last five weeks, all of our differential equations have been autonomous. Now we turn to second-order equations that model systems that are subject to some type of external forcing. Here are two examples:

Example. The nonlinear pendulum with a pivot point that is subject to vertical oscillations. The motion of such a pendulum is governed by the second-order nonlinear equation

$$m\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + k\sin\theta = F\sin\theta\cos\omega t$$

where ω determines the frequency of the oscillations of the pivot point and F determines the amplitude of the oscillations. The `Pendulums` tool on the CD illustrates this system.

Example. The linear mass-spring system where the spring is subject to vertical oscillations. To model this system, we use the standard mass-spring system and add a term that corresponds to the force added to the system by the oscillations. We get

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F\cos\omega t.$$

The `ForcedMassSpring` tool on the CD illustrates this system.

In class we will discuss forced linear equations only, but your second project will involve some experimentation with the forced pendulum.

Our success studying unforced linear systems was due in large part to the Linearity Principle. For forced linear equations, we are fortunate to have the Extended Linearity Principle.

Extended Linearity Principle Consider a nonhomogeneous equation (a forced equation)

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t)$$

and its corresponding homogeneous equation (the unforced equation)

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0.$$

1. Suppose $y_p(t)$ is a particular solution of the nonhomogeneous equation and $y_h(t)$ is a solution of the corresponding homogeneous equation. Then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the corresponding homogeneous equation.

Therefore, if $k_1 y_1(t) + k_2 y_2(t)$ is the general solution of the homogeneous equation, then

$$k_1 y_1(t) + k_2 y_2(t) + y_p(t)$$

is the general solution of the nonhomogeneous equation.

This principle provides the basic framework that we will use to solve linear second-order forced equations. (At this point in the course, you should go back and review the method described in Section 1.8 for solving nonhomogeneous first-order linear equations.)

We already know how to find the general solution to the associated homogeneous equation, so we need only find one solution to the original equation.

Example. Consider the equation

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = e^{3t}.$$