

More on undamped sinusoidal forcing

Last class we considered the following example.

Example.

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t$$

For $\omega \neq \pm\sqrt{3}$, we obtained the general solution

$$y(t) = k_1 \cos \sqrt{3}t + k_2 \sin \sqrt{3}t + \frac{1}{3 - \omega^2} \cos \omega t,$$

and we concentrated on the initial condition $(y(0), y'(0)) = (0, 0)$. The solution to this initial-value problem is

$$y(t) = \frac{1}{3 - \omega^2} (\cos \omega t - \cos \sqrt{3}t).$$

What happens if $\omega = \sqrt{3}$?

Example.

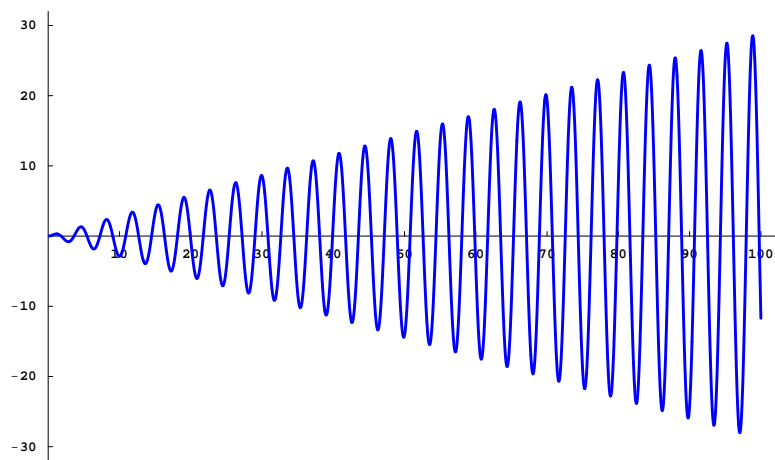
$$\frac{d^2y}{dt^2} + 3y = \cos \sqrt{3}t$$

The complexified equation is

$$\frac{d^2y}{dt^2} + 3y = e^{i\sqrt{3}t}.$$

What guess should we use?

Here is the graph of $y_p(t)$.



Linearization:

We would like to apply what we know about linear systems to nonlinear systems.

Example. Consider the van der Pol equation

$$\frac{d^2x}{dt^2} + (x^2 - 1)\frac{dx}{dt} + x = 0.$$

The corresponding system is

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= (1 - x^2)y - x.\end{aligned}$$

Let's calculate the equilibria:

Example. Consider the (undamped) pendulum

$$\frac{d^2\theta}{dt^2} + \sin \theta = 0.$$

The corresponding system is

$$\begin{aligned}\frac{d\theta}{dt} &= v \\ \frac{dv}{dt} &= -\sin \theta.\end{aligned}$$

Let's calculate the equilibria:

The linearized system near $(\pi, 0)$ is

Given the (nonlinear) system

$$\begin{aligned}\frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y),\end{aligned}$$

its **Jacobian** at the point (x_0, y_0) is the matrix

$$\mathbf{J}(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

and its linearization at (x_0, y_0) is the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}\mathbf{Y}.$$

For the pendulum, we have one linearization for each equilibrium point:

For the van der Pol equation, we obtain the linearization:

Linearization Theorem Let \mathbf{Y}_0 be an equilibrium point for the nonlinear autonomous system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$$

and let

$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}\mathbf{Y}$$

be the corresponding linearized system. If the eigenvalues of \mathbf{J} are not purely imaginary, then the solution curves of the nonlinear system near \mathbf{Y}_0 behave in the same qualitative way as the solution curves of the linear system.

Example. Consider the van der Pol equation near the origin. The linearized system is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{Y}.$$

