A little bit more on linearization

Linearization Theorem Let $\mathbf{Y}_0$ be an equilibrium point for the nonlinear autonomous system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$$

and let

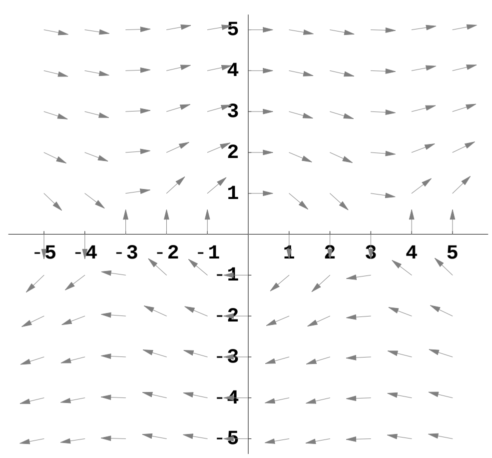
$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}\mathbf{Y}$$

be the corresponding linearized system. If the eigenvalues of $\mathbf{J}$ are not purely imaginary, then the solution curves of the nonlinear system near $\mathbf{Y}_0$ behave in the same qualitative way as the solution curves of the linear system.

Example. Consider the pendulum equation. The linearized system near $(0, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$

This example is misleading.

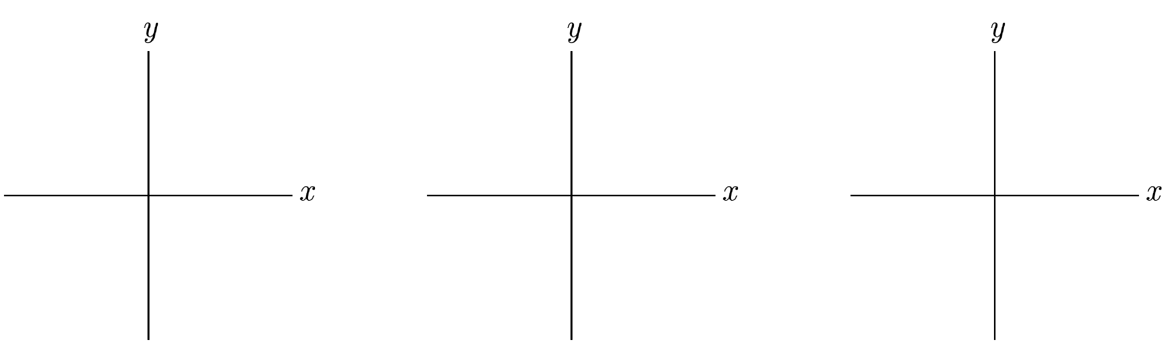


What is special about the case of purely imaginary eigenvalues in the linearization?

Example. Consider the one-parameter family of systems

$$\begin{aligned}\frac{dx}{dt} &= -y + \alpha x(x^2+y^2)\\ \frac{dy}{dt} &= \quad x + \alpha y(x^2+y^2)\end{aligned}$$

where α is a parameter. Note that $(0,0)$ is always an equilibrium point.



$\alpha < 0$ $\alpha = 0$ $\alpha > 0$

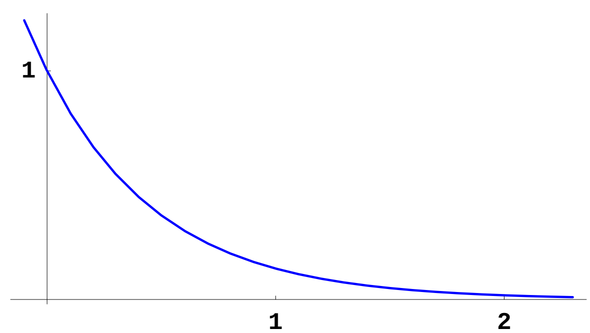
The Laplace transform

For the remainder of the semester, we are going to take a somewhat different approach to the solution of differential equations. We are going to study a way of transforming differential equations into algebraic equations.

We begin with a little review of improper integrals.

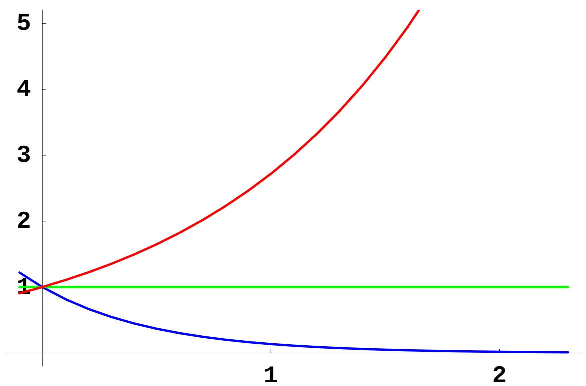
Example. Consider the improper integral

$$\int_0^\infty e^{-2t}\, dt.$$



Example. Consider the improper integrals

$$\int_0^\infty e^{-st}\,dt.$$



Definition. The *Laplace transform* of the function $y(t)$ is the function

$$Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt.$$

This transform is an "operator" (a function on functions). It transforms the function $y(t)$ into the function $Y(s)$.

Notation: We often represent this operator using the script letter $\mathcal{L}$. In other words,

$$\mathcal{L}[y] = Y.$$

For example,

$$\mathcal{L}[1] = \frac{1}{s}.$$

Note that, even if $y(t)$ is defined for all t, the Laplace transform $Y(s)$ may not be defined for all s.

Examples. Using *Mathematica* to calculate the improper integrals, we see that:

$$\mathcal{L}[\sin t] = \frac{1}{s^2+1} \quad \text{for} \quad s > 0,$$

$$\mathcal{L}[e^{2t} \sin 3t] = \frac{3}{s^2 - 4s + 13} \quad \text{for} \quad s > 2$$

$$\mathcal{L}[t^4] = \frac{24}{s^5} \quad \text{for} \quad s > 0$$

$$\mathcal{L}[t \cos \sqrt{2}\, t] = \frac{s^2 - 2}{(s^2+2)^2} \quad \text{for} \quad s > 0$$

$$\mathcal{L}[e^{i\omega t}] = \frac{1}{s - i\omega} \quad \text{for} \quad s > 0$$

Example. Let's compute $\mathcal{L}[e^{at}]$ using the definition and the improper integrals we have already computed:

Properties of the Laplace transform There are two properties of the Laplace transform that make it well suited for solving linear differential equations:

1. $\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$
2. $\mathcal{L}$ is a linear transform

Both of these properties are extremely important, but the surprising one is #1. Let's consider

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^\infty \left(\frac{dy}{dt}\right) e^{-st}\,dt.$$

In fact, before we consider the improper integral, let's apply the method of integration by parts to the indefinite integral

$$\int \left(\frac{dy}{dt}\right) e^{-st}\,dt.$$

Now let's see how we can use the Laplace transform to solve an initial-value problem.

Example. Solve the IVP

$$\frac{dy}{dt} - 3y = e^{2t}, \quad y(0) = 4.$$

1. Transform both sides of the equation:

2. Solve for $\mathcal{L}[y]$:

3. Calculate the inverse Laplace transform: