More on the Laplace transform

Last class we defined the Laplace transform.

Definition. The *Laplace transform* of the function $y(t)$ is the function

$$Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt.$$

This transform is an "operator" (a function on functions). It transforms the function $y(t)$ into the function $Y(s)$.

Notation: We often represent this operator using the script letter $\mathcal{L}$. In other words,

$$\mathcal{L}[y] = Y.$$

For example,

$$\mathcal{L}[1] = \frac{1}{s}, \quad \mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad \text{and} \quad \mathcal{L}[\sin t] = \frac{1}{s^2+1}.$$

Note that even if $y(t)$ is defined for all t, the Laplace transform $Y(s)$ may not be defined for all s.

Properties of the Laplace transform There are two properties of the Laplace transform that make it well suited for solving linear differential equations:

1. $\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$
2. $\mathcal{L}$ is a linear transform

Both of these properties are extremely important, but the surprising one is #1. Let's consider

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^\infty \left(\frac{dy}{dt}\right) e^{-st}\, dt.$$

In fact, before we consider the improper integral, let's apply the method of integration by parts to the indefinite integral

$$\int \left(\frac{dy}{dt}\right) e^{-st}\, dt.$$

Now let's see how we can use the Laplace transform to solve an initial-value problem.

Example. Solve the IVP

$$\frac{dy}{dt} - 3y = e^{2t}, \quad y(0) = 4.$$

1. Transform both sides of the equation:

2. Solve for $\mathcal{L}[y]$:

3. Calculate the inverse Laplace transform:

Is this the right answer? Do we need Laplace transforms to calculate it?

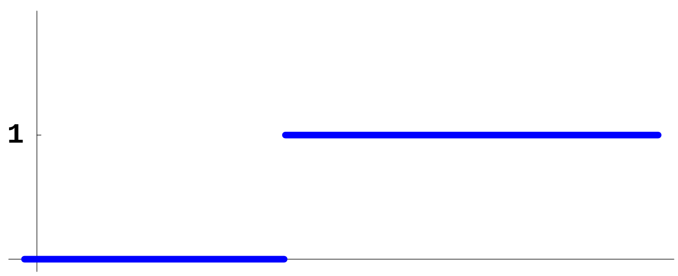
Discontinuous differential equations

The Laplace transform helps solve linear differential equations that are discontinuous in one way or another.

Definition. The *Heaviside function* $u_a(t)$ is the function defined by

$$u_a(t) = \begin{cases} 0, & \text{if } t < a; \\ 1, & \text{if } t \geq a. \end{cases}$$

Thus $u_a(t)$ has a discontinuity at $t = a$ where it jumps from 0 to 1.



Here's how you can use the Heaviside function to avoid piecewise definitions:

Example. Consider $g(t) = 2t + u_1(t)(2 - 2t)$.

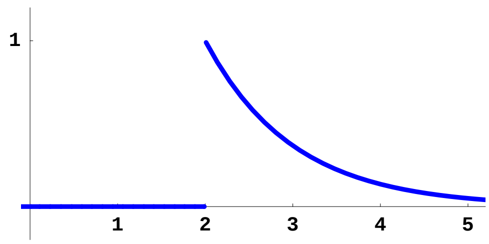
How do we calculate the Laplace transform of a discontinuous function?

Example. Let's calculate $\mathcal{L}[u_a]$ directly from the definition of $\mathcal{L}$.

In order to calculate inverse Laplace transforms in this situation, we need another property of the transform.

Rule 3: Shifting the t-axis. $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}\mathcal{L}[f]$.

Example. Calculate $\mathcal{L}[g]$ where $g(t) = u_2(t)\, e^{-(t-2)}$.



Why does the shifting rule work the way that it does?

Shifting the t-axis. Consider $\mathcal{L}[u_a(t) f(t-a)]$.

Now let's see how we can use these properties of the Laplace transform to solve an initial-value problem that involves discontinuous forcing.

Example. Solve the IVP

$$\frac{dv}{dt} + v = u_2(t), \quad v(0) = 3.$$