More on Laplace transforms and second-order equations

The second-order equations that we discussed last class were undamped. In order to consider the full range of second-order equations, we need one more property of the transform.

**Shifting the $s$-axis.** Let $Y(s)$ denote $\mathcal{L}[y(t)]$. Then

$$\mathcal{L}[e^{at}\,y(t)] =$$

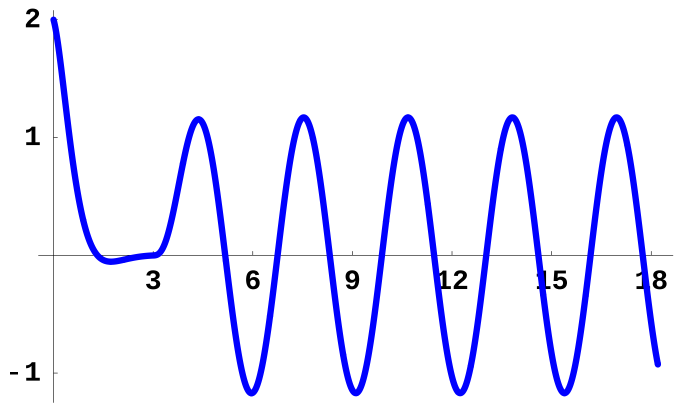
**Example 1.** Calculate $\mathcal{L}[e^{-2t}\cos 3t]$.

**Example 2.** Calculate $\mathcal{L}^{-1}\left[\dfrac{2s+7}{s^2+4s+7}\right]$.

Let's solve the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 7y = 10\,u_3(t)\ \sin 2(t-3), \quad y(0) = 2,\ y'(0) = -1.$$

Before we get too far into the messy formulas, let's look at the graph of the solution using `HPGSystemSolver`:




Now for the formulas:

1. Transform both sides of the equation:

2. Solve for $\mathcal{L}[y]$:

3. Calculate the inverse Laplace transform:

We calculated

$$\mathcal{L}^{-1}\left[\frac{2s+7}{s^2+4s+7}\right] = 2e^{-2t}\cos\sqrt{3}\,t + \sqrt{3}e^{-2t}\sin\sqrt{3}\,t$$

in Example 2.

To invert the second term, we take advantage of some algebra done before class:

(a) Partial fractions decomposition:

$$\frac{1}{(s^2+4)(s^2+4s+7)} = \frac{1}{73}\left(\frac{4s+13}{s^2+4s+7} - \frac{4s-3}{s^2+4}\right)$$

(b) Inverse related to the first term:

$$\mathcal{L}^{-1}\left[\frac{4s+13}{s^2+4s+7}\right] = 4e^{-2t}\cos\sqrt{3}\,t + \frac{5\sqrt{3}}{3}e^{-2t}\sin\sqrt{3}\,t$$

(c) Inverse related to the second term:

$$\mathcal{L}^{-1}\left[\frac{4s-3}{s^2+4}\right] = 4\cos 2t - \frac{3}{2}\sin 2t$$

After we put all of this together, we get the solution

$$y(t) = 2e^{-2t}\cos\sqrt{3}\,t + \sqrt{3}e^{-2t}\sin\sqrt{3}\,t +$$

$$\frac{20}{73}\,u_3(t)\left(4e^{-2(t-3)}\cos\sqrt{3}(t-3) + \frac{5\sqrt{3}}{3}e^{-2(t-3)}\sin\sqrt{3}(t-3) - 4\cos 2(t-3) + \frac{3}{2}\sin 2(t-3)\right)$$

Summary of transform rules and table of standard transforms

Here are important selections from the summary on page 620 in your text.

| $y(t)$ | $Y(s) = \mathcal{L}[y]$ | |
|---|---|---|
| $y(t) = 1$ | $Y(s) = \frac{1}{s}$ | $(s > 0)$ |
| $y(t) = e^{at}$ | $Y(s) = \frac{1}{s-a}$ | $(s > a)$ |
| $y(t) = u_a(t)$ | $Y(s) = \frac{e^{-as}}{s}$ | $(s > 0)$ |
| $y(t) = \cos \omega t$ | $Y(s) = \frac{s}{s^2+\omega^2}$ | $(s > 0)$ |
| $y(t) = \sin \omega t$ | $Y(s) = \frac{\omega}{s^2+\omega^2}$ | $(s > 0)$ |

Properties of the Laplace Transform

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

$$\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2]$$

$$\mathcal{L}[\alpha\, y] = \alpha\mathcal{L}[y] \text{ for any constant } \alpha$$

$$\mathcal{L}[u_a(t)y(t-a)] = e^{-as}\mathcal{L}[y]$$

$$\mathcal{L}[e^{at}y(t)] = Y(s-a) \text{ where } Y = \mathcal{L}[y]$$

Some people like to memorize a few more entries such as

$$\mathcal{L}[e^{at}\cos\omega t] = \frac{s-a}{(s-a)^2+\omega^2},$$

but I prefer to use the last rule (shifting the $s$-axis). Also, the rule for $\mathcal{L}[dy/dt]$ in terms of $\mathcal{L}[y]$ yields

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2\mathcal{L}[y] - y(0)s - y'(0).$$

**Warning:** Just because you can solve a linear differential equation with the Laplace transform does not mean that you should forget what you learned in previous parts of the course. The transform method is particularly well suited for differential equations with discontinuous forcing.