A little more on the vector field of a system

Last class we saw that the system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

can be written in vector form as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$$

where

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad \text{and} \qquad \mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}.$$

Here's another example:

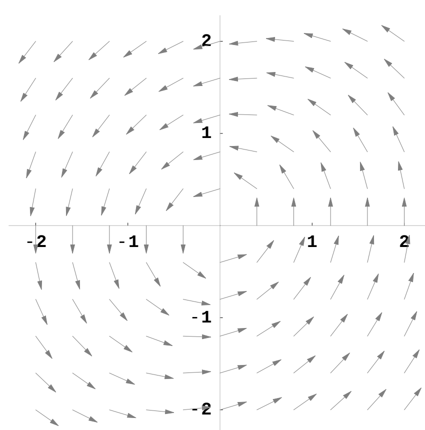
Example 2. Consider the system

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x - 0.3y.$$

The vector field associated with this system is

$$\mathbf{F}(\mathbf{Y}) = \mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x - 0.3y \end{pmatrix}.$$

Here's the direction field:



On page 7 of yesterday's handout, there was a matching problem from an old exam. There's also a matching problem in this week's homework. Doing these problems is a good way to make sure that you understand how the system of differential equations determines a vector field.

Analytic techniques:

There are few analytic techniques that work for both linear and nonlinear systems.

1. You can always check to see if a given function is a solution (no wrong answers).

   For example, consider the initial-value problem

$$\frac{dx}{dt} = 2y - x \qquad (x_0, y_0) = (2, 1).$$
$$\frac{dy}{dt} = y$$

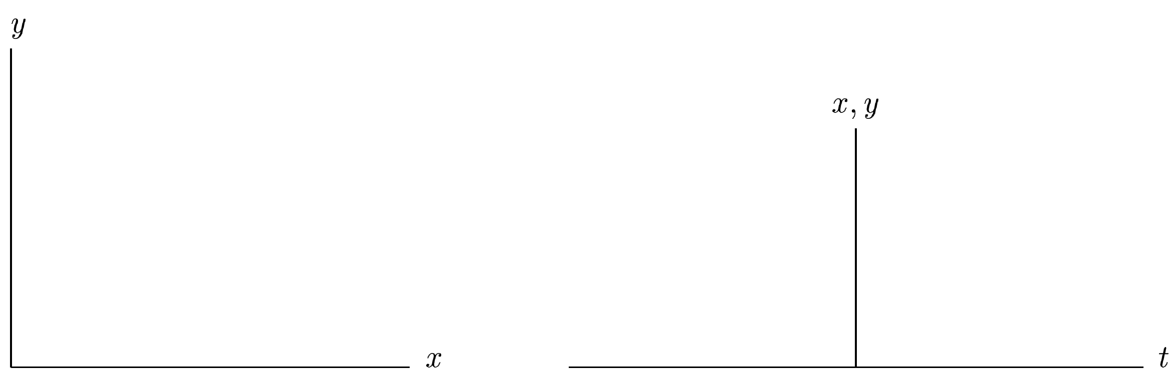
   Using the vector notation

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

   we can write this IVP as

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2y - x \\ y \end{pmatrix}, \qquad \mathbf{Y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

   First, let's see what the solution looks like when we graph it with `HPGSystemSolver`:



Claim: The function

$$\mathbf{Y}(t) = \begin{pmatrix} e^t + e^{-t} \\ e^t \end{pmatrix}$$

solves the IVP.

2. General solution of a partially-decoupled system

Example. Consider the previous system

$$\frac{dx}{dt} = 2y - x$$
$$\frac{dy}{dt} = y.$$

We can calculate the general solution using methods we learned for first-order equations:

(More blank space at the top of the next page.)

Damped harmonic oscillator

Let's return to our mass-spring system and add a term that models damping.

Assumption: The damping force is proportional to the speed of the mass and it acts as a restoring force.

The second-order equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

and its equivalent system

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{k}{m}y - \frac{b}{m}v\end{aligned}$$

appear in many applications. On the CD, you will find it in `MassSpring` and `RLCCircuits`, and it has also been used to study biological processes such as the blood glucose regulatory system in humans.

There is a guessing technique for the damped harmonic oscillator

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

Example. Consider the harmonic oscillator

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0.$$

Its characteristic equation is

You should plot these solutions with `HPGSystemSolver`. What are the corresponding solution curves and component graphs?