

## Linear systems

Last class we started to discuss linear systems, that is, the ones that can be written as

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned} \quad \text{or as} \quad \frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{Y}$$

where  $a$ ,  $b$ ,  $c$ , and  $d$  are constants. These constants are also referred to as the coefficients or as the parameters of the system.

Last class we also reviewed two examples that we had discussed previously.

**Example 1.** We have already calculated the general solution to the partially decoupled system

$$\begin{aligned} \frac{dx}{dt} &= 2y - x \\ \frac{dy}{dt} &= y. \end{aligned}$$

Written in vector notation, the general solution is

$$\mathbf{Y}(t) = e^t \begin{pmatrix} y_0 \\ y_0 \end{pmatrix} + e^{-t} \begin{pmatrix} x_0 - y_0 \\ 0 \end{pmatrix}.$$

**Example 2.** For the damped harmonic oscillator

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0,$$

we used a guessing technique to find the two (scalar) solutions  $y_1(t) = e^{-t}$  and  $y_2(t) = e^{-2t}$ . As usual, this second-order equation can be converted to a first-order system where

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -2y - 3v, \end{aligned}$$

and the two scalar solutions yield two vector solutions

$$\mathbf{Y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Today we will learn that once we have these two solutions in Example 2 we know the general solution.

---

---

Given a linear system  $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$ , how do we calculate the vector in the vector field at any given point  $\mathbf{Y}_0$ ?

How do we calculate the equilibrium points of  $\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$ ?

**Example.** Let  $\mathbf{A}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ .

**Example.** Let  $\mathbf{A}_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ .

**Theorem.** The origin is always an equilibrium point of a linear system. It is the only equilibrium point if and only if  $\det \mathbf{A} \neq 0$ .

## The Linearity Principle

Let's return to Example 1. For practice, we'll use vector notation this time:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

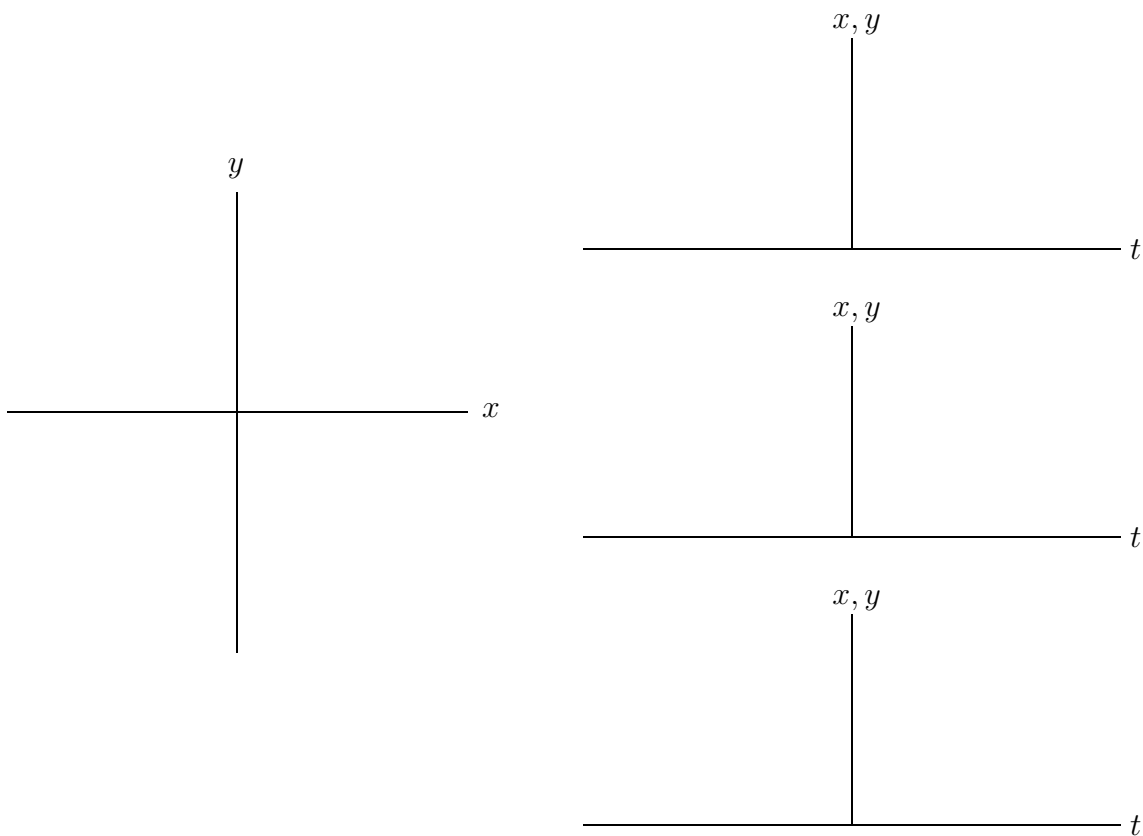
Also consider three different initial conditions

$$\mathbf{Y}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{Y}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{Y}_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

They correspond to the three solutions

$$\mathbf{Y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_2(t) = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{Y}_3(t) = \begin{pmatrix} e^t + e^{-t} \\ e^t \end{pmatrix}.$$

Let's see what happens when we graph these solutions.



How are these three solutions related?

**Linearity Principle** Suppose

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

is a linear system of differential equations.

1. If  $\mathbf{Y}(t)$  is a solution of this system and  $k$  is any constant, then  $k\mathbf{Y}(t)$  is also a solution.
2. If  $\mathbf{Y}_1(t)$  and  $\mathbf{Y}_2(t)$  are two solutions of this system, then  $\mathbf{Y}_1(t) + \mathbf{Y}_2(t)$  is also a solution.

This principle gives us a more general way to find solutions of linear systems. To see how this approach works, let's consider Example 1 again along with the two solutions  $\mathbf{Y}_1(t)$  and  $\mathbf{Y}_2(t)$ .

**Example.** Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

and the two solutions

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

Any linear combination of  $\mathbf{Y}_1(t)$  and  $\mathbf{Y}_2(t)$  is also a solution to the system.

**Example.** Solve

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

In general, how many solutions do we need to be able to solve any initial-value problem?