Straight-line solutions

To solve a linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

where $\mathbf{A}$ is a 2×2 matrix, we use the Linearity Principle applied to two *linearly independent* solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$. The general solution is

$$k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t).$$

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

and the two solutions

$$\mathbf{Y}_1(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We can solve any initial-value problem for this differential equation using an appropriate linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$. In other words, the general solution of this system is

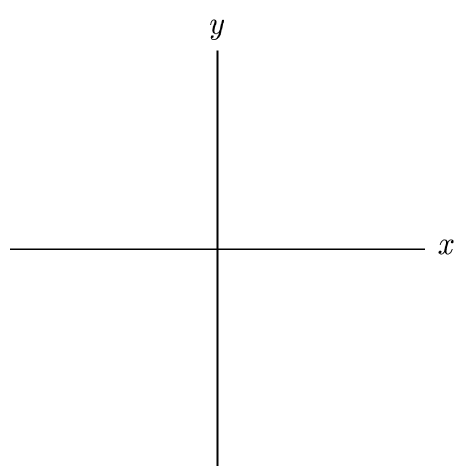
$$k_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note the difference between this version of the general solution and the general solution

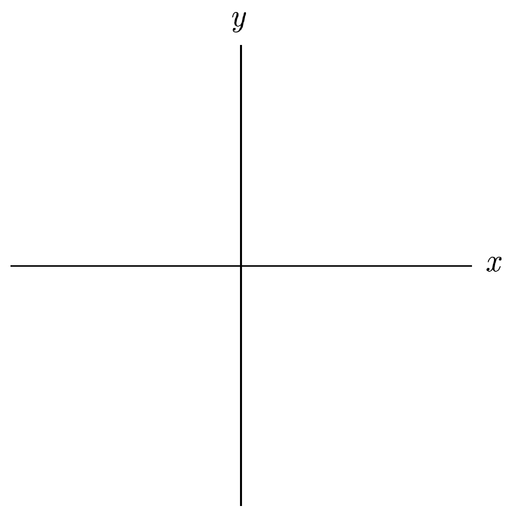
$$\mathbf{Y}(t) = e^{-t}\begin{pmatrix} x_0 - y_0 \\ 0 \end{pmatrix} + e^{t}\begin{pmatrix} y_0 \\ y_0 \end{pmatrix},$$

which we obtained by solving this system as a partially decoupled system.

Questions: How do we find two linearly independent solutions? Is there something special about the two solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ in the example?



For a general linear system of the form $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$, what geometric property of the vector field guarantees the existence of these "straight-line" solutions?



"Straight-line" Solutions. Suppose that

$$\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$$

for some nonzero vector $\mathbf{Y}_0$ and some scalar λ. Then the function

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$$

is a solution to the linear differential equation $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$.

We want nonzero initial conditions $\mathbf{Y}_0$ (vectors) so that

$$\mathbf{A}\mathbf{Y}_0 = \lambda \mathbf{Y}_0$$

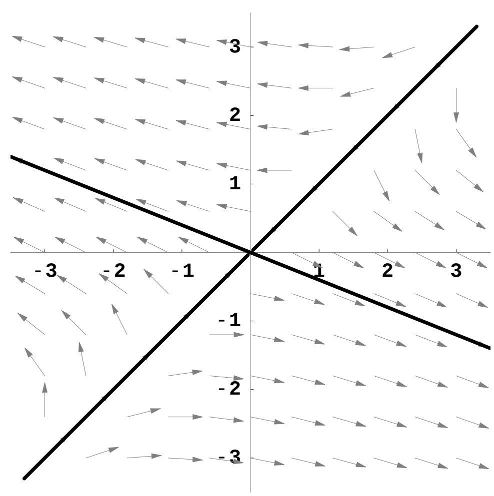
for some scalar λ.

Terminology: The scalar λ is called an *eigenvalue* of the matrix $\mathbf{A}$ and the vector $\mathbf{Y}_0$ is called an *eigenvector* associated to the eigenvalue λ.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$

First let's see what `MatrixFields` tells us about the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.



Aside from the theory of algebraic linear equations

For what matrices $\mathbf{B}$ does the equation $\mathbf{BY} = \mathbf{0}$ have nontrivial solutions?

Singular Matrices. The matrix equation

$$\mathbf{BY} = \mathbf{0}$$

has nontrivial solutions $\mathbf{Y}$ if and only if $\det \mathbf{B} = 0$.

Note: Most matrices are nonsingular.

Now let's use this theorem to find eigenvalues and eigenvectors:

Example. Find the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$