MA 226 March 17, 2008

Linear systems—a brief review

A linear system (with constant coefficients) can be written as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

where $\mathbf{A}$ is a square matrix of constants (the coefficients). For us, $\mathbf{A}$ will be a 2×2 matrix. Using the Linearity Principle, we can produce many solutions from just a few:

> If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, then
>
> $$k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$$
>
> is a solution for any choice of constants k_1 and k_2.

“Straight-line” Solutions. Suppose that

$$\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$$

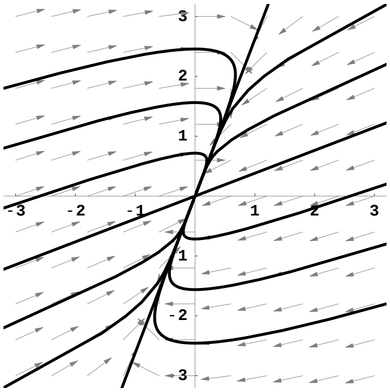
for some nonzero vector $\mathbf{Y}_0$ and some scalar λ. Then the function

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$$

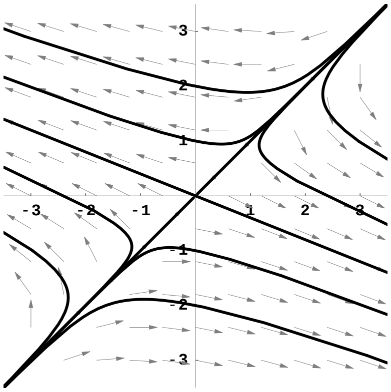
is a solution to the linear differential equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}.$$

Summary for real and distinct (nonzero) eigenvalues



sink ($\lambda_1 < \lambda_2 < 0$)



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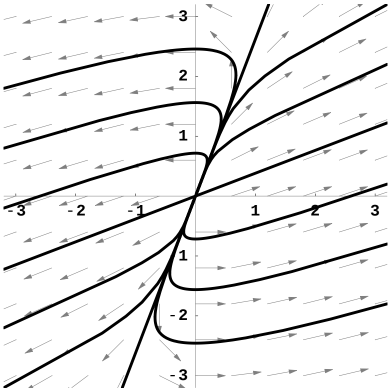


source ($0 < \lambda_1 < \lambda_2$)

Complex eigenvalues

Use Euler's formula

$$e^{(a+bi)t} = e^{at}\left(\cos bt + i \sin bt\right)$$

applied to complex-valued "straight-line" solutions.

Theorem. Consider $d\mathbf{Y}/dt = \mathbf{AY}$, where $\mathbf{A}$ is a matrix with real entries. If $\mathbf{Y}_c(t)$ is a complex-valued solution, then both

$$\mathrm{Re}\mathbf{Y}_c(t) \quad \text{and} \quad \mathrm{Im}\mathbf{Y}_c(t)$$

are real-valued solutions, and they are linearly independent.

Just before spring break, I was in the middle of describing three examples.

Example 1. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

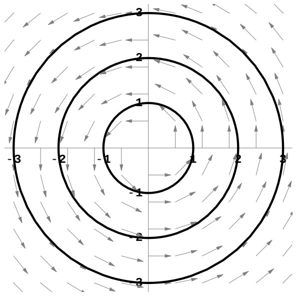
The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 + 1$, so the eigenvalues are $\lambda = \pm i$. One eigenvector associated to the eigenvalue $\lambda = i$ is

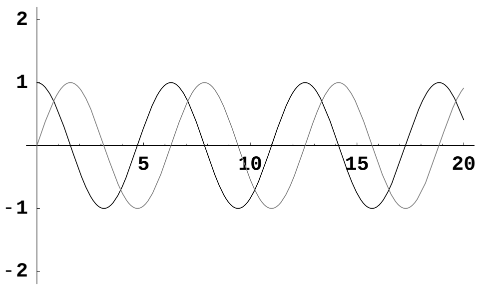
$$\mathbf{Y}_0 = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

We ended up with a general solution of the form

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + k_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

We get circles centered at the origin in the phase plane.





Example 2. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{BY}$ where

$$\mathbf{B} = \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{B}$ is $\lambda^2 + 4$, so the eigenvalues are $\lambda = \pm 2i$. One eigenvector associated to the eigenvalue $\lambda = 2i$ is

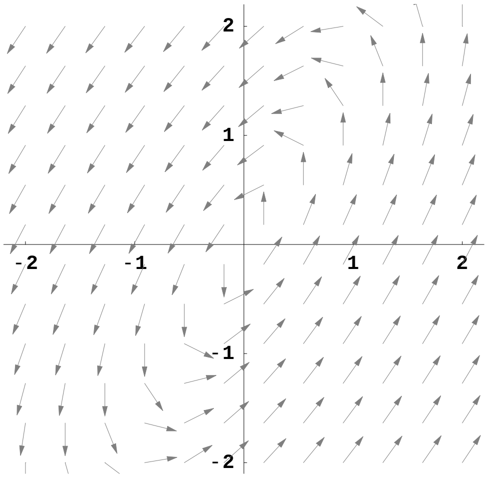
$$\mathbf{Y}_0 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$

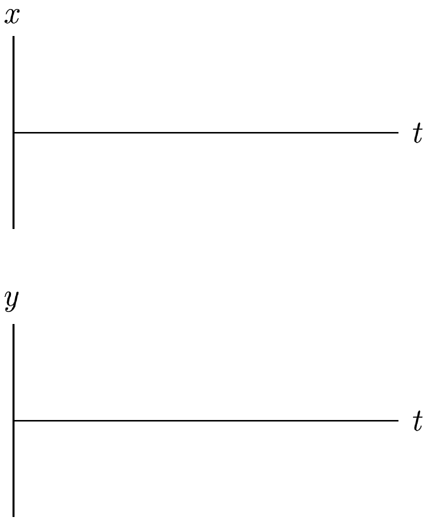
By taking the real and imaginary parts of the complex-valued solution

$$\begin{aligned} \mathbf{Y}_c(t) &= e^{(2t)i} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \\ &= (\cos 2t + i \sin 2t) \begin{pmatrix} 1+i \\ 2 \end{pmatrix}, \end{aligned}$$

we ended up with a general solution of the form

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} \cos 2t - \sin 2t \\ 2\cos 2t \end{pmatrix} + k_2 \begin{pmatrix} \cos 2t + \sin 2t \\ 2 \sin 2t \end{pmatrix}.$$



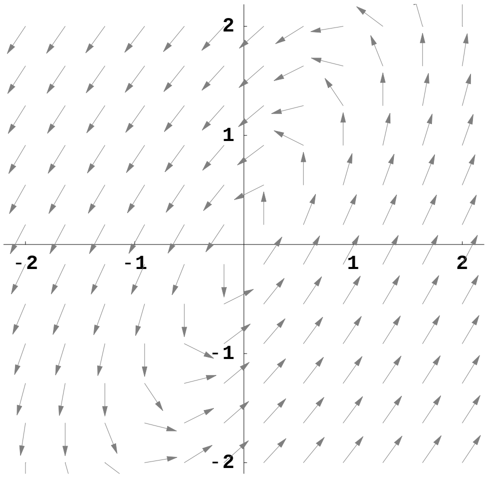


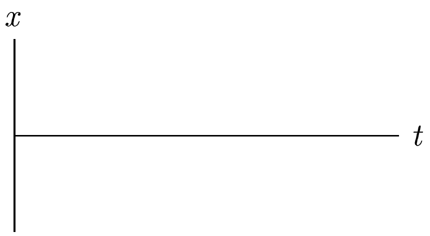
Example 3. $\dfrac{d\mathbf{Y}}{dt} = \mathbf{C}\mathbf{Y}$ where

$$\mathbf{C} = \begin{pmatrix} 1.9 & -2 \\ 4 & -2.1 \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{C}$ is $\lambda^2 + 0.2\lambda + 4.01$, so the eigenvalues are $\lambda = -0.1 \pm 2i$. One eigenvector associated to the eigenvalue $\lambda = -0.1 + 2i$ is

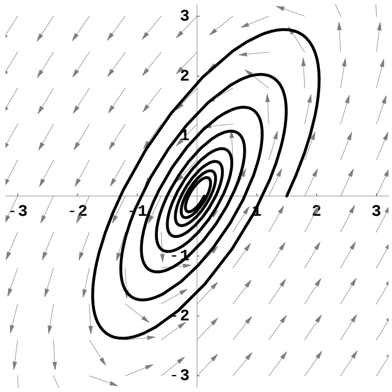
$$\mathbf{Y}_0 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$



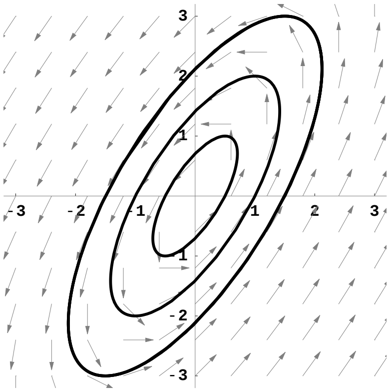


Summary: Linear systems with complex eigenvalues $\lambda = a \pm bi$

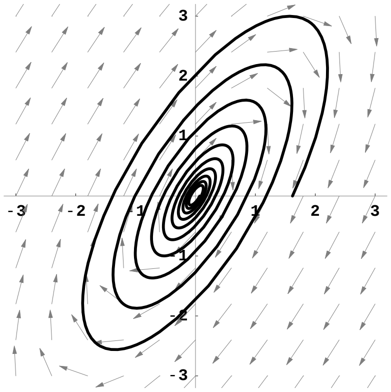
Here are the possible phase portraits:



spiral sink ($a < 0$)



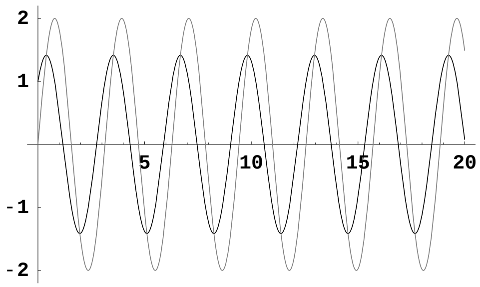
center ($a = 0$)



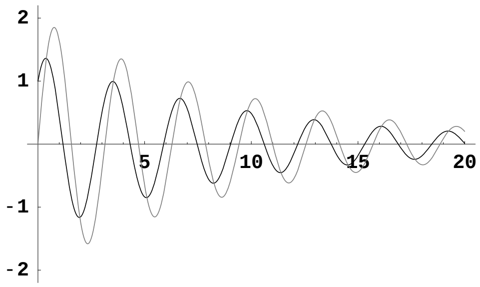
spiral source ($a > 0$)

What information can you get just from the complex eigenvalue alone?

Recall Example 2. The eigenvalues are $\lambda = \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:



In Example 3, the eigenvalues are $\lambda = -0.1 \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:



Frequency versus period: The solutions in Example 3 are not periodic in the strict sense. There is no time T such that

$$x(t+T) = x(t) \quad \text{and} \quad y(t+T) = y(t)$$

for all t. However, there is a period associated to these solutions. In the text, we call this the **natural period** of the solutions.

Perhaps it is best to think about these solutions as oscillating solutions that are decaying over time and to measure the oscillations in terms of their **frequency**.

Definition. The *frequency* F of an oscillating function $g(t)$ is the number of cycles that $g(t)$ makes in one unit of time.

Suppose that $g(t)$ is oscillating periodically with "period" T. What is its frequency F?

Example. Consider the standard sinusoidal functions $g(t) = \cos \beta t$ and $g(t) = \sin \beta t$.

Suppose we measure frequency in radians rather than in cycles. This measure of frequency is often called **angular frequency**. Let's denote the angular frequency by f. Then

$$f = 2\pi F.$$

Repeated eigenvalues: Sometimes the characteristic polynomial has the same real root twice. When this happens, we say that the eigenvalues are "repeated."

Example. $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{A}$ is $(\lambda - 2)^2$, so there is only one eigenvalue, $\lambda = 2$. Let's calculate the associated eigenvectors:

But we already know how to solve this system. How?