More on repeated eigenvalues

Suppose $\mathbf{A}$ is a typical $2 \times 2$ matrix with a repeated eigenvalue $\lambda$. Then last class we guessed that its solutions would be of the form

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}_0 + te^{\lambda t}\mathbf{V}_1,$$

and when we tested this guess, we learned that if

$$\mathbf{V}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{V}_0,$$

then $\mathbf{Y}(t)$ is a solution to the system. If $\mathbf{V}_0$ is an eigenvector, then $\mathbf{V}_1 = \mathbf{0}$. If not, then $\mathbf{V}_1$ is an eigenvector.

Sinks with repeated eigenvalues: If $\lambda < 0$, then $\mathbf{Y}(t) \to \mathbf{0}$ as $t \to \infty$. Also, we saw that

$$\text{direction}(\mathbf{Y}(t)) \to \text{direction}(\mathbf{V}_1)$$

as $t \to \infty$.

Last class we computed the general solution to the following example with the repeated eigenvalue $\lambda = -2$.

**Example.** $\dfrac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}.$$

The general solution is

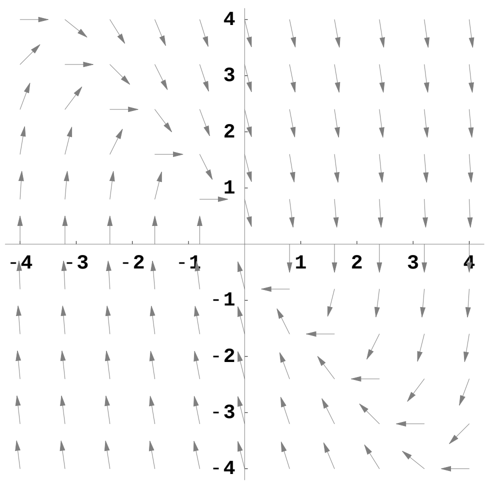
$$\mathbf{Y}(t) = e^{-2t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-2t}\begin{pmatrix} 2x_0 + y_0 \\ -4x_0 - 2y_0 \end{pmatrix}.$$

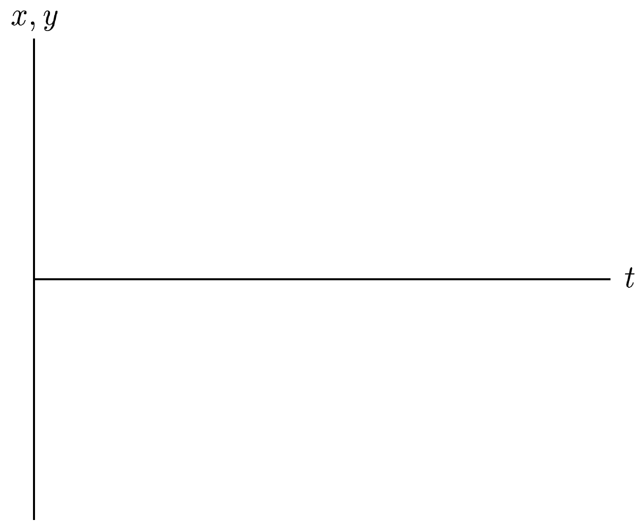
It is interesting to look at the example

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}\mathbf{Y}$$

using two of the tools on the CD. Using `LinearPhasePortraits`, we can see that this system is on the boundary between spiral sinks and real sinks.

We can also use HPGSystemSolver to plot the phase portrait and a typical pair of $x(t)$- and $y(t)$-graphs.



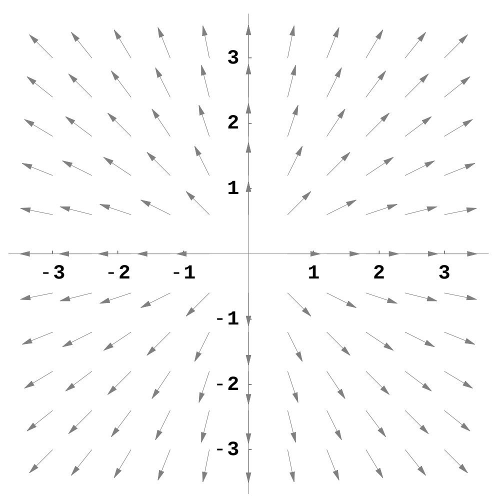




Unusual case of repeated eigenvalues: There is one type of linear system that has repeated eigenvalues that is different from the examples we have discussed.

**Example.** Consider $d\mathbf{Y}/dt = \mathbf{AY}$ where $\mathbf{A}$ is the diagonal matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

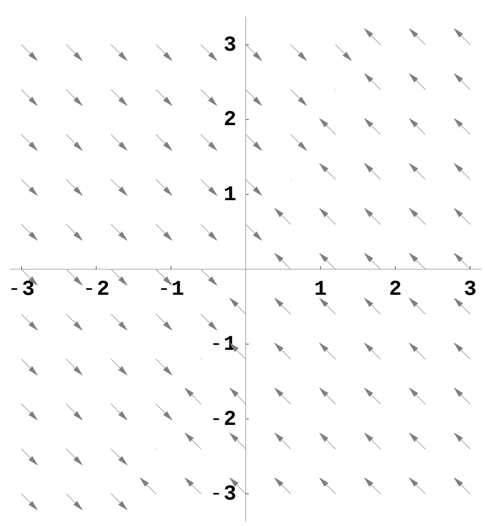
What are its eigenvalues and eigenvectors?



Finally consider the example

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \mathbf{Y}.$$

Its characteristic polynomial is $\lambda^2 + 3\lambda$. So its eigenvalues are $\lambda = -3$ and $\lambda = 0$. (If a system has 0 as an eigenvalue, we say that it is *degenerate*. The matrix $\mathbf{A}$ of coefficients is singular—see class notes for February 29.)



Second-order, linear equations: We will apply what we have learned about linear systems to solve second-order homogeneous linear equations.

Let's return to the guessing technique for second-order equations that we learned about a month ago (February 20 and 22) and see how it relates to what we have done with linear systems recently.

**Example.** Consider the equation

$$2\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 4y = 0.$$

1. Use a guessing technique to find two nonzero solutions $y_1(t)$ and $y_2(t)$ that are not multiples of each other.

2. Convert this equation to a first-order system and determine the analogous solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.

3. In what way are $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ special solutions?

Let's see how this guessing technique can be used to solve all second-order homogeneous equations.

Consider

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

with its characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

as well as the corresponding system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{c}{a}\,y - \frac{b}{a}\,v \end{aligned}$$

with its characteristic equation

$$\det \begin{pmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{pmatrix} = 0.$$

Useful observation: If $\lambda$ is an eigenvalue, the vector

$$\mathbf{Y}_0 = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

is *always* an associated eigenvector.