Homogeneous linear second-order equations

Last class we revisited the guessing technique (guess $y(t) = e^{\lambda t}$) for equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

(see Section 2.3 and notes for February 20 and 22). Let's see how this guessing technique can be used to solve all homogeneous linear second-order equations.

Consider the second-order differential equation with its characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

as well as the corresponding system

$$\begin{aligned}\frac{dy}{dt} &= v\\ \frac{dv}{dt} &= -\frac{c}{a}y - \frac{b}{a}v\end{aligned}$$

with its characteristic equation

$$\det\begin{pmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{pmatrix} = 0.$$

Useful observation: If λ is an eigenvalue, the vector $\mathbf{Y}_0 = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$ is *always* an associated eigenvector.

Let's see what that observation tells us about solutions to the second-order equation. There are three cases:

1. Two real, distinct, nonzero eigenvalues λ_1 and λ_2:

2. A complex-conjugate pair of eigenvalues $\lambda = \alpha \pm i\beta$, with $\beta \neq 0$:

3. One nonzero real eigenvalue λ of multiplicity two:

Conclusion: We can determine the general solution of a homogeneous linear second-order equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

immediately from the characteristic equation $a\lambda^2 + b\lambda + c = 0$.

YOU DO NOT NEED TO CALCULATE THE EIGENVECTORS OR EVEN REDUCE TO A FIRST-ORDER SYSTEM if you simply want to produce the general solution of a linear second-order equation.

Application: We can apply what we have learned to the (damped) harmonic oscillator

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

In this case, we are assuming that the parameters m and k are positive and that $b \geq 0$. The characteristic equation $m\lambda^2 + b\lambda + k = 0$ has eigenvalues

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are three cases based on the value of the discriminant $b^2 - 4mk$.

1. $b^2 - 4mk < 0$

2. $b^2 - 4mk = 0$

3. $b^2 - 4mk > 0$

We can see the progression from underdamped to critically damped to overdamped with a Quicktime animation I have posted on the web site.

The trace, the determinant, and the characteristic polynomial

The characteristic polynomial of a 2×2 matrix can be expressed in terms of its trace and determinant.

Consider the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let's calculate the characteristic polynomial of $\mathbf{A}$:

Conclusion: The eigenvalues of any 2×2 matrix are determined by the trace and the determinant of $\mathbf{A}$. We have

$$\lambda = \frac{(\mathrm{tr}\mathbf{A}) \pm \sqrt{(\mathrm{tr}\mathbf{A})^2 - 4(\det \mathbf{A})}}{2}.$$