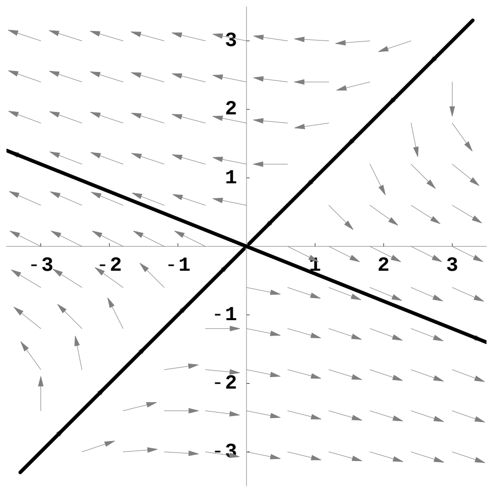
Eigenvalues, eigenvectors, and straight-line solutions

To find the straight-line solutions of the linear system $d\mathbf{Y}/dt = \mathbf{AY}$, we need to find the eigenvalues and associated eigenvectors of the matrix $\mathbf{A}$. We find the eigenvalues using the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

Example. Find the general solution to $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}$.

Using HPGSystemSolver, we plot the phase portrait for this system.



Facts about eigenvalues and eigenvectors: Given a 2×2 matrix $\mathbf{A}$,

1. The characteristic equation can have two real roots, one real root of multiplicity two, or two complex conjugate roots.

2. Given an eigenvector $\mathbf{Y}_0$ associated to an eigenvalue λ, then any nonzero scalar multiple $\mathbf{Y}_0$ is also an eigenvector associated to λ.

3. Eigenvectors associated to distinct eigenvalues are linearly independent.

Summary of Case of Two Distinct Real Eigenvalues

Suppose $\mathbf{A}$ is a matrix with two eigenvalues λ_1 and λ_2. To be consistent, we will assume that $\lambda_1 < \lambda_2$, that $\mathbf{V}_1$ is an eigenvector associated to λ_1, and that $\mathbf{V}_2$ is an eigenvector associated to λ_2. The general solution of

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

is

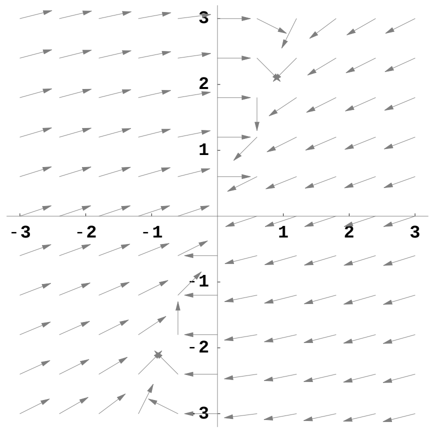
$$\mathbf{Y}(t) = k_1 e^{\lambda_1 t}\mathbf{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2.$$

Case 1: $\lambda_1 < \lambda_2 < 0$.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$

(Direction field on top of next page.)



Sketching component graphs

Once we understand the phase portrait, we should also be able to sketch the component graphs without `HPGSystemSolver`.

For example, once again consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$

Let's sketch the $x(t)$- and $y(t)$-graphs that correspond to the initial conditions $(-3, 2)$ and $(3, 2)$.