A little more on linearization

Example. Consider the equation

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0$$

for the undamped pendulum. The corresponding system is

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -\sin\theta.$$

The linearized system near $(\pi, 0)$ is

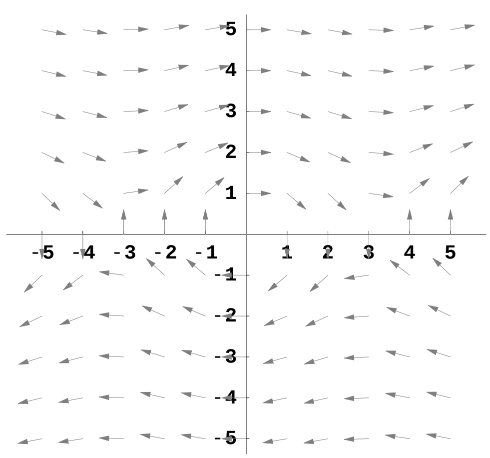
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{Y}.$$

Its characteristic equation is $\lambda^2 - 1 = 0$, and therefore the eigenvalues are ± 1. The Linearization Theorem says that this equilibrium point is a nonlinear saddle.

The linearized system near $(0, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y},$$

and the characteristic equation is $\lambda^2 + 1 = 0$. The eigenvalues are $\pm i$. This equilibrium point is a nonlinear center, but this example is misleading. The Linearization Theorem does not apply to this equilibrium point.

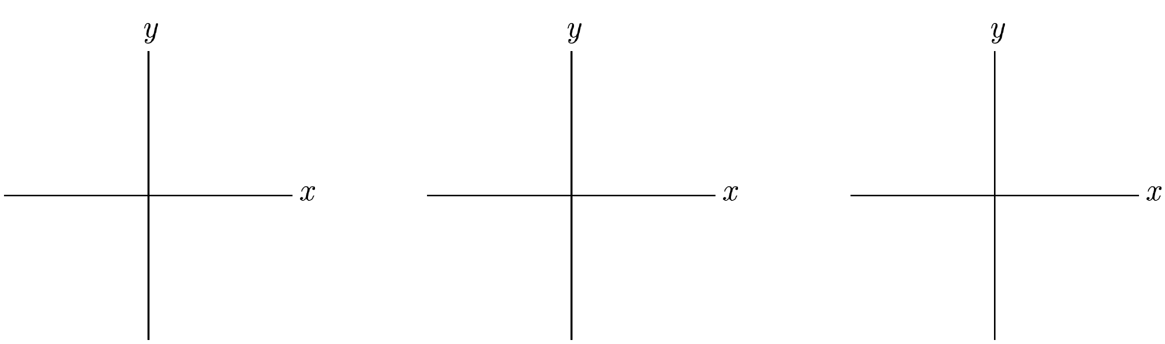


What is special about the case of purely imaginary eigenvalues in the linearization?

Example. Consider the one-parameter family of systems

$$\begin{aligned}\frac{dx}{dt} &= -y + \alpha x(x^2 + y^2)\\ \frac{dy}{dt} &= \quad x + \alpha y(x^2 + y^2)\end{aligned}$$

where α is a parameter. Note that $(0,0)$ is always an equilibrium point.



$\alpha < 0$ $\qquad$ $\alpha = 0$ $\qquad$ $\alpha > 0$

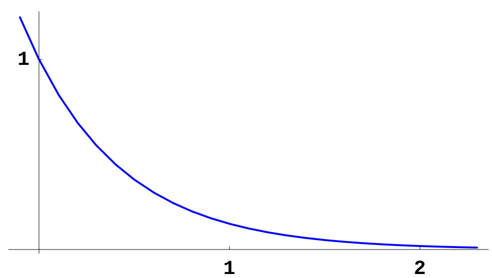
The Laplace transform

For the remainder of the semester, we are going to take a somewhat different approach to the solution of differential equations. We are going to study a way of transforming differential equations into algebraic equations.

We begin with a little review of improper integrals.

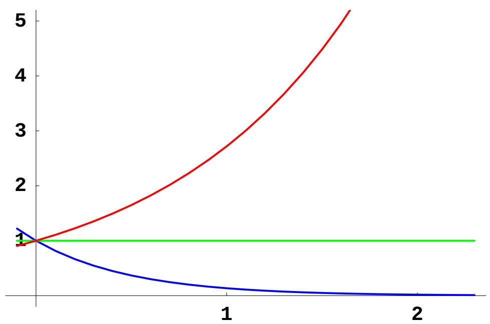
Example. Consider the improper integral

$$\int_0^\infty e^{-2t}\,dt.$$



Example. Consider the improper integrals

$$\int_0^\infty e^{-st}\,dt.$$



Definition. The *Laplace transform* of the function $y(t)$ is the function

$$Y(s) = \int_0^\infty y(t)\,e^{-st}\,dt.$$

This transform is an "operator" (a function on functions). It transforms the function $y(t)$ into the function $Y(s)$.

Notation: We often represent this operator using the script letter $\mathcal{L}$. In other words,

$$\mathcal{L}[y] = Y.$$

For example, $\mathcal{L}[1] = \frac{1}{s}$.

Note that, even if $y(t)$ is defined for all t, the Laplace transform $Y(s)$ may not be defined for all s.

Example. Let's compute $\mathcal{L}[e^{at}]$ using the definition and the improper integrals we have already computed:

Examples. Using *Mathematica* to calculate the improper integrals, we see that:

$$\mathcal{L}[\sin t] = \frac{1}{s^2+1} \quad \text{for} \quad s > 0,$$

$$\mathcal{L}[e^{2t}\sin 3t] = \frac{3}{s^2-4s+13} \quad \text{for} \quad s > 2$$

$$\mathcal{L}[t^4] = \frac{24}{s^5} \quad \text{for} \quad s > 0$$

$$\mathcal{L}[\sin 2t] = \frac{2}{s^2+4} \quad \text{for} \quad s > 0,$$

$$\mathcal{L}[t\,\cos\sqrt{2}\,t] = \frac{s^2-2}{(s^2+2)^2} \quad \text{for} \quad s > 0$$

$$\mathcal{L}[e^{i\omega t}] = \frac{1}{s-i\omega} \quad \text{for} \quad s > 0$$