Discontinuous differential equations

Last class we discussed the Laplace transform method of solving linear differential equations. Today we see how this transform helps solve equations that are discontinuous in one way or another.

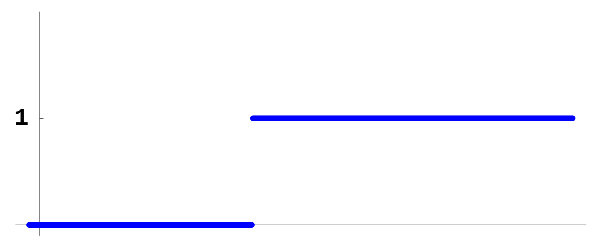
Recall that the Laplace transform of $y(t)$ is the function $Y(s)$ given by

$$Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt.$$

Definition. The *Heaviside function* $u_a(t)$ is the function defined by

$$u_a(t) = \begin{cases} 0, & \text{if } t < a; \\ 1, & \text{if } t \geq a. \end{cases}$$

Thus $u_a(t)$ has a discontinuity at $t = a$ where it jumps from 0 to 1.



Here's how you can use the Heaviside function to avoid piecewise definitions:

Example. Consider $g(t) = 2t + u_1(t)(2 - 2t)$.

Laplace transforms are very convenient if we have discontinuous forcing. Remember the process for solving differential equations using Laplace transforms:

1. Transform both sides of the differential equation.

2. Determine $\mathcal{L}[y]$.

3. Compute the inverse Laplace transform of $\mathcal{L}[y]$.

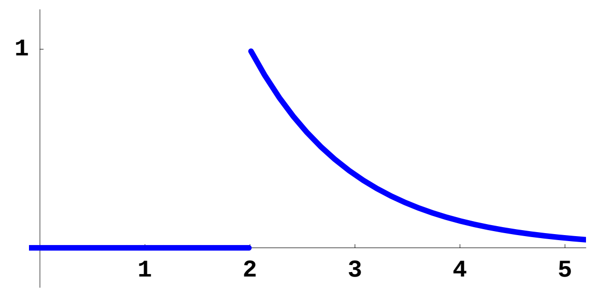
How do we calculate the Laplace transform of a discontinuous function?

Example. Let's calculate $\mathcal{L}[u_a]$ directly from the definition of $\mathcal{L}$.

In order to calculate inverse Laplace transforms, we need another property of the transform.

Rule 3: Shifting the t-axis. $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}\mathcal{L}[f]$.

Example. Calculate $\mathcal{L}[g]$ where $g(t) = u_2(t)\, e^{-(t-2)}$.



Why does the shifting rule work the way that it does?

Shifting the t-axis. Let's compute

$\mathcal{L}[u_a(t)f(t-a)] =$

Now let's see how we can use these properties of the Laplace transform to solve an initial-value problem that involves discontinuous forcing.

Example. Solve the IVP

$$\frac{dv}{dt} + v = u_2(t), \quad v(0) = 3.$$

1. Transform both sides of the equation:

2. Solve for $\mathcal{L}[v]$:

3. Calculate the inverse Laplace transform:

Now let's plot this solution using `HPGSolver`.