

Using the Laplace transform to solve certain second-order equations

Last Friday we computed the two inverse Laplace transforms

$$\mathcal{L}^{-1} \left[\frac{2s + 1}{s^2 + 9} \right] = 2 \cos 3t + \frac{1}{3} \sin 3t$$

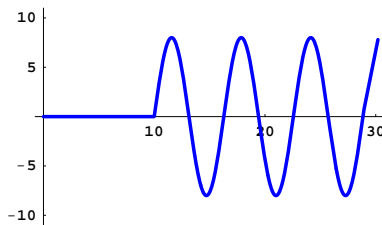
and

$$\mathcal{L}^{-1} \left[\frac{8e^{-10s}}{(s^2 + 9)(s^2 + 1)} \right] = u_{10}(t) \left(\sin(t - 10) - \frac{1}{3} \sin 3(t - 10) \right).$$

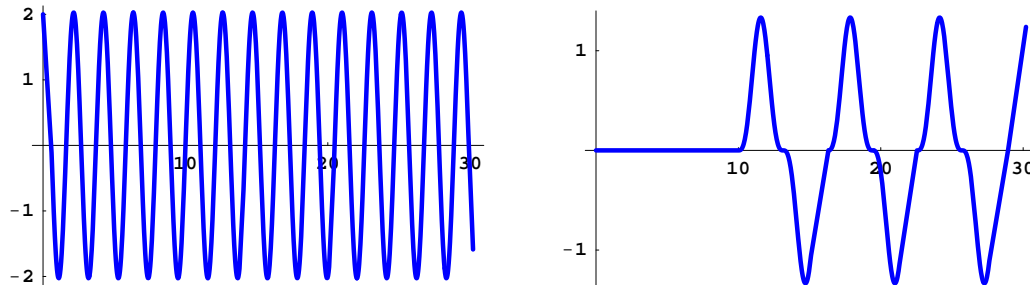
Now let's use them to solve the initial-value problem

$$\frac{d^2y}{dt^2} + 9y = 8u_{10}(t) \sin(t - 10), \quad y(0) = 2, \quad y'(0) = 1.$$

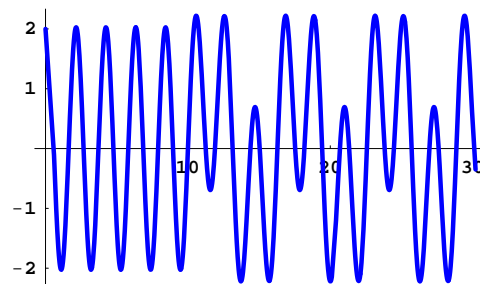
Here is a graph of the forcing function $8u_{10}(t) \sin(t - 10)$.



Here are the graphs of the two functions that combine to give us the desired solution.



Here is the graph of the solution



That second-order equation is undamped. In order to consider the full range of second-order equations, we need one more property of the transform.

Shifting the s -axis. Let $Y(s)$ denote the Laplace transform $\mathcal{L}[y(t)]$. Then

$$\mathcal{L}[e^{at} y(t)] =$$

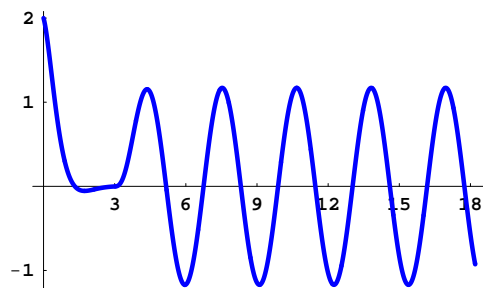
Example 1. Calculate $\mathcal{L}[e^{-2t} \cos 3t]$.

Example 2. Calculate $\mathcal{L}^{-1} \left[\frac{2s + 7}{s^2 + 4s + 7} \right]$.

Let's solve the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 7y = 10u_3(t) \sin 2(t - 3), \quad y(0) = 2, \quad y'(0) = -1.$$

Before we get too far into the messy formulas, let's look at the graph of the solution using `HPGSystemSolver`:



Now for the formulas:

1. Transform both sides of the equation:

2. Solve for $\mathcal{L}[y]$:

3. Calculate the inverse Laplace transform:

We calculated

$$\mathcal{L}^{-1} \left[\frac{2s + 7}{s^2 + 4s + 7} \right] = 2e^{-2t} \cos \sqrt{3}t + \sqrt{3}e^{-2t} \sin \sqrt{3}t$$

in Example 2.

To invert the second term, we take advantage of some algebra done before class:

(a) Partial fractions decomposition:

$$\frac{1}{(s^2 + 4)(s^2 + 4s + 7)} = \frac{1}{73} \left(\frac{4s + 13}{s^2 + 4s + 7} - \frac{4s - 3}{s^2 + 4} \right)$$

(b) Inverse related to the first term:

$$\mathcal{L}^{-1} \left[\frac{4s + 13}{s^2 + 4s + 7} \right] = 4e^{-2t} \cos \sqrt{3}t + \frac{5\sqrt{3}}{3}e^{-2t} \sin \sqrt{3}t$$

(c) Inverse related to the second term:

$$\mathcal{L}^{-1} \left[\frac{4s - 3}{s^2 + 4} \right] = 4 \cos 2t - \frac{3}{2} \sin 2t$$

After we put all of this together, we get the solution

$$y(t) = 2e^{-2t} \cos \sqrt{3}t + \sqrt{3}e^{-2t} \sin \sqrt{3}t + \frac{20}{73} u_3(t) \left(4e^{-2(t-3)} \cos \sqrt{3}(t-3) + \frac{5\sqrt{3}}{3}e^{-2(t-3)} \sin \sqrt{3}(t-3) - 4 \cos 2(t-3) + \frac{3}{2} \sin 2(t-3) \right)$$