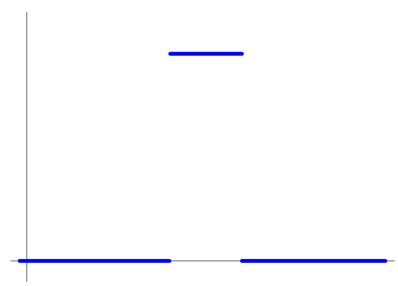
Dirac Delta Function

The Dirac delta "function" $\delta_a(t)$ is used to model impulse forcing. In other words, suppose we want to model a unit force that is applied instantaneously at time $t = a$. We begin with the function

$$g_{\Delta t}(t) = \begin{cases} \dfrac{1}{\Delta t}, & \text{if } a - \dfrac{\Delta t}{2} \leq t < a + \dfrac{\Delta t}{2}; \\ 0, & \text{otherwise.} \end{cases}$$



We can write $g_{\Delta t}$ in terms of the Heaviside function. We get

$$g_{\Delta t}(t) = \frac{1}{\Delta t}\left(u_{a-\frac{\Delta t}{2}}(t) - u_{a+\frac{\Delta t}{2}}(t)\right).$$

Now let's calculate the Laplace transform of $g_{\Delta t}$:

We take the limit as $\Delta t \to 0$.

Dirac Delta Function. The Dirac delta function $\delta_a(t)$ is the "function" such that

$$\mathcal{L}[\delta_a] = e^{-as}.$$

Example. Consider the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 7y = \delta_5(t), \quad y(0) = 2,\ y'(0) = -1.$$

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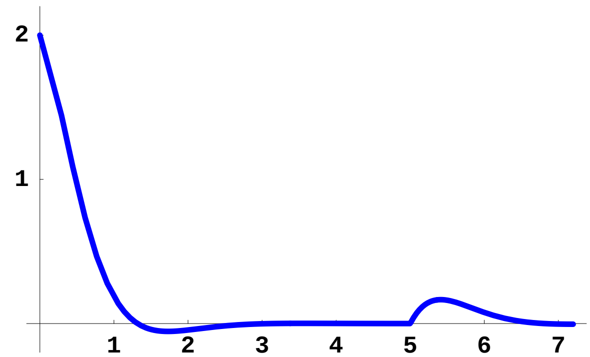
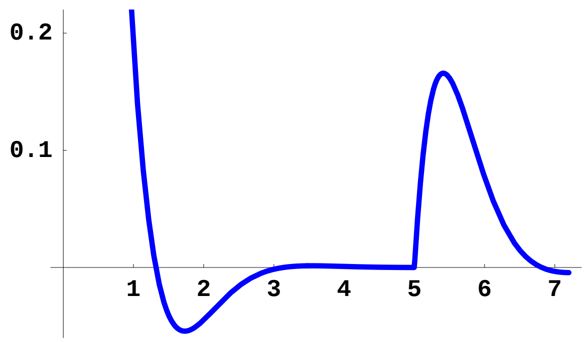
Here is the graph of the solution:



We enlarge the scale on the vertical axis:



Here is the graph of its derivative: