One more example that uses the Laplace transform

Let's solve one more initial-value problem using the Laplace transform. We will simplify the solution by doing some preliminary calculations.

Example 1. Calculate $\mathcal{L}^{-1}\left[\dfrac{s+4}{s^2+2s+5}\right]$.

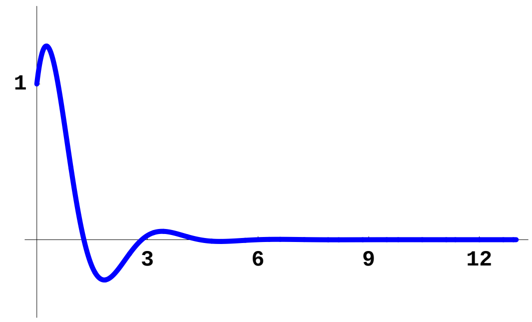
Example 2. Calculate $\mathcal{L}^{-1}\left[\dfrac{e^{-3s}}{s^2+2s+5}\right]$.

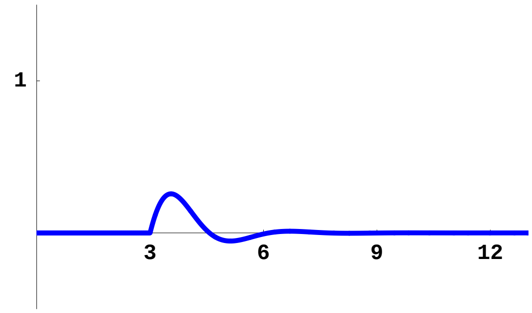
Example 3. Calculate $\mathcal{L}^{-1}\left[\dfrac{e^{-6s}}{s(s^2+2s+5)}\right]$.

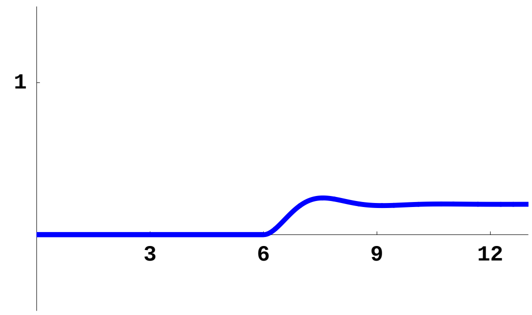
Example. Solve the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \delta_3(t) + u_6(t), \quad y(0) = 1, \quad y'(0) = 2.$$

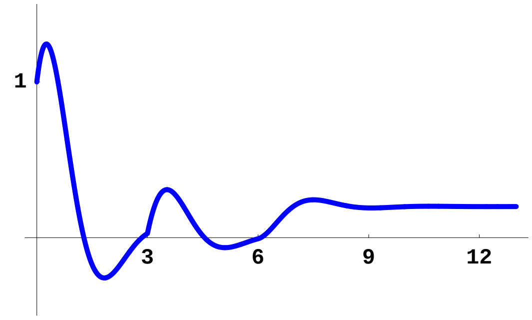
Here are graphs of the three functions that combine to make up the solution:







Here is the graph of the solution:



Summary of transform rules and table of standard transforms

Here are important selections from the summary on page 620 in your text.

$y(t)$	$Y(s) = \mathcal{L}[y]$	
$y(t) = 1$	$Y(s) = \dfrac{1}{s}$	$(s > 0)$
$y(t) = e^{at}$	$Y(s) = \dfrac{1}{s-a}$	$(s > a)$
$y(t) = u_a(t)$	$Y(s) = \dfrac{e^{-as}}{s}$	$(s > 0)$
$y(t) = \cos \omega t$	$Y(s) = \dfrac{s}{s^2+\omega^2}$	$(s > 0)$
$y(t) = \sin \omega t$	$Y(s) = \dfrac{\omega}{s^2+\omega^2}$	$(s > 0)$
$y(t) = \delta_a(t)$	$Y(s) = e^{-as}$	

Properties of the Laplace Transform

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

$$\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2]$$

$$\mathcal{L}[\alpha\, y] = \alpha\mathcal{L}[y] \text{ for any constant } \alpha$$

$$\mathcal{L}[u_a(t)y(t-a)] = e^{-as}\mathcal{L}[y]$$

$$\mathcal{L}[e^{at}y(t)] = Y(s-a) \text{ where } Y = \mathcal{L}[y]$$

Some people like to memorize a few more entries such as

$$\mathcal{L}[e^{at}\cos\omega t] = \frac{s-a}{(s-a)^2+\omega^2},$$

but I prefer to use the last rule (shifting the s-axis). Also, the rule for $\mathcal{L}[dy/dt]$ in terms of $\mathcal{L}[y]$ yields

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2\mathcal{L}[y] - y(0)s - y'(0).$$

Warning: Just because you can solve a linear differential equation with the Laplace transform does not mean that you should forget what you learned in previous parts of the course. The transform method is particularly well suited for differential equations with discontinuous and impulse forcing.