Last class we did sinusoidal forcing with damping. Today we discuss sinusoidal forcing in the absense of damping.

A little more on the steady-state solution

First, there is one piece of unfinished business from Wednesday's class. On Wednesday, we calculated the steady-state solution

$$y_p(t) = -\tfrac{1}{4}(\cos 2t - \sin 2t)$$

for the equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = \cos 2t,$$

and we did so using computations that involved complex numbers. In fact, we found $y_p(t)$ as the real part of

$$y_c(t) = -\tfrac{1}{4}(1+i)\, e^{(2i)t}.$$

At the end of class, I was in the middle of justifying why the complex number

$$a = -\tfrac{1}{4}(1+i)$$

tells us everything we need to know about the steady-state solution.

Using polar coordinates in the complex plane (see pp. 745–747 in Appendix C), we see that

$$a = -\tfrac{1}{4}(1+i) = \tfrac{\sqrt{2}}{4}\, e^{i(-\frac{3}{4}\pi)}.$$

What does this polar representation of a tell us about the steady-state solution?

Sinusoidal forcing in the absense of damping

Now consider the mass-spring system without the dashpot.

Example. Let's find the general solution to

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t.$$

Note the lack of a damping term. We want to see what happens with various forcing frequencies.

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Unfortunately the parts of the solution that correspond to the associated homogeneous equation do not die out. So to get some qualitative understanding in this case, we make a simplifying assumption. We consider the solution that satisfies the initial condition $(y(0), y'(0)) = (0, 0)$.

On the web site, there is a Quicktime animation of the graphs of these solutions as we vary the forcing frequency ω. The following trig identity helps us interpret what we see in the animation.

Trig identity:

$$\cos at - \cos bt = -2\,(\sin \alpha t)\,(\sin \beta t)$$

where

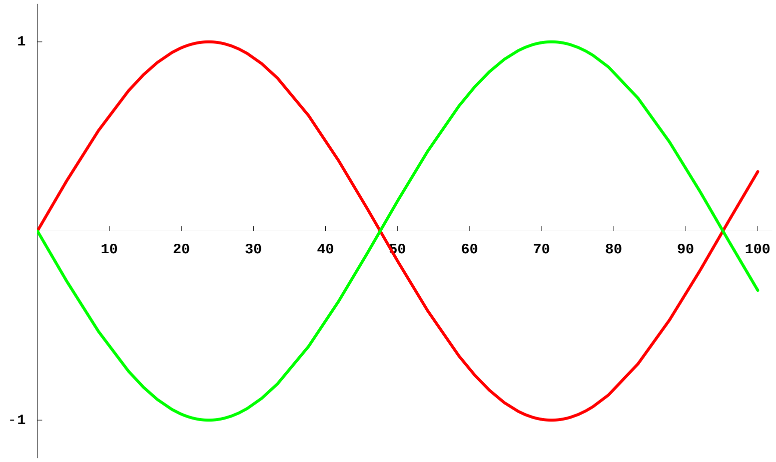
$$\alpha = \frac{a+b}{2} \quad \text{and} \quad \beta = \frac{a-b}{2}.$$

The number α is the average of a and b, and β is called the *half-difference* of a and b.

Example. Let's use this trig identity to get a rough idea of the graph of

$$\cos \omega t - \cos \sqrt{3}\, t$$

where $\omega = 1.6$.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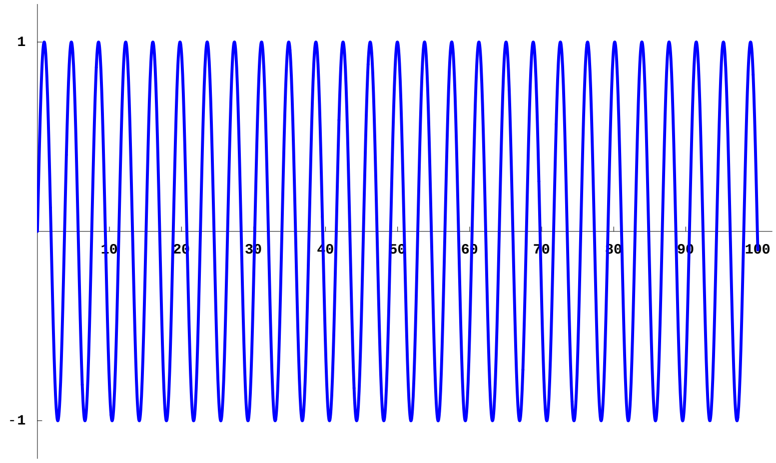
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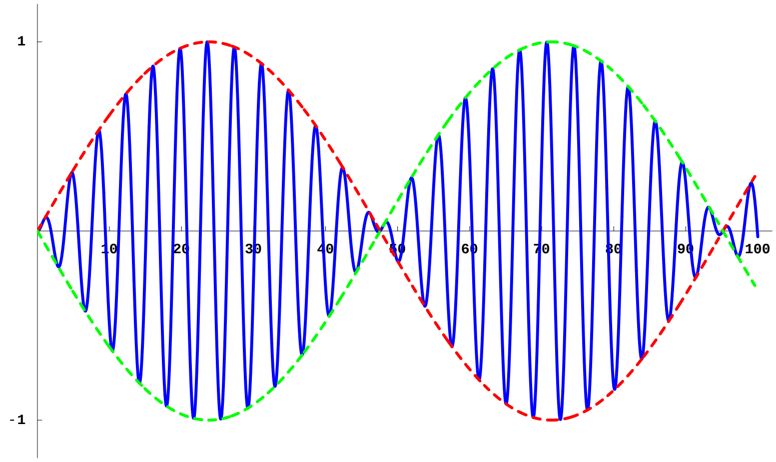
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Let's return to the solution to

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t$$

that satisfies the initial condition $(y(0), y'(0)) = (0, 0)$. If $\omega \neq \pm\sqrt{3}$, the solution is

$$y(t) = \frac{1}{3-\omega^2}(\cos \omega t - \cos \sqrt{3}\, t).$$

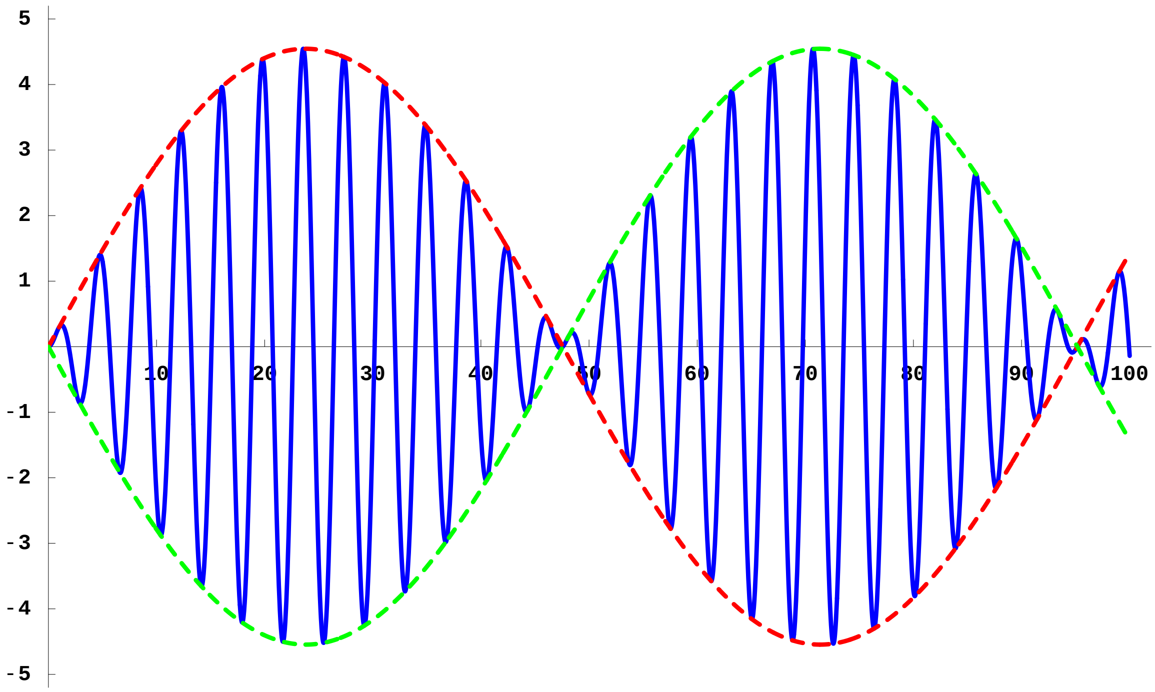
Applying the trig identity, we obtain

$$y(t) = \frac{-2}{3-\omega^2}\ (\sin \alpha t)\,(\sin \beta t)$$

where

$$\alpha = \frac{\omega + \sqrt{3}}{2} \quad \text{and} \quad \beta = \frac{\omega - \sqrt{3}}{2}.$$

Here is the graph of this solution in the case where $\omega = 1.6$.



What happens if $\omega = \sqrt{3}$?