More on undamped sinusoidal forcing

Last class we computed the general solution to

$$\frac{d^2y}{dt^2} + 3y = \cos\omega t$$

assuming $\omega^2 \neq 3$, and we got $y(t) = k_1 \cos\sqrt{3}\,t + k_2 \sin\sqrt{3}\,t + \dfrac{1}{3-\omega^2}\cos\omega t$.

Then, in order to understand the behavior of the solutions, we restricted our attention to those solutions that satisfy the initial condition $(y(0), y'(0)) = (0, 0)$. Solving for k_1 and k_2 yields

$$y(t) = \frac{1}{3-\omega^2}(\cos\omega t - \cos\sqrt{3}\,t).$$

On the web site, there is a Quicktime animation of the graphs of these solutions as we vary the forcing frequency ω. The following trig identity helps us interpret what we see in the animation.

Trig identity:

$$\cos at - \cos bt = -2\,(\sin\alpha t)\,(\sin\beta t)$$

where

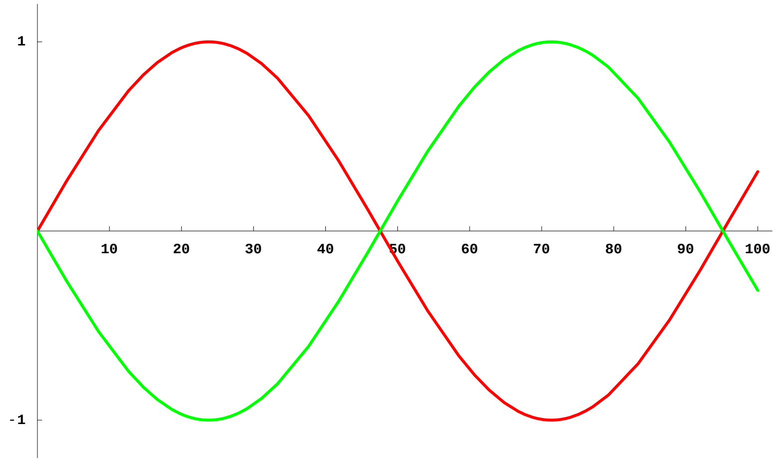
$$\alpha = \frac{a+b}{2} \quad \text{and} \quad \beta = \frac{a-b}{2}.$$

The number α is the average of a and b, and β is called the *half-difference* of a and b.

Example. Let's use this trig identity to get a rough idea of the graph of

$$\cos\omega t - \cos\sqrt{3}\,t$$

where $\omega = 1.6$.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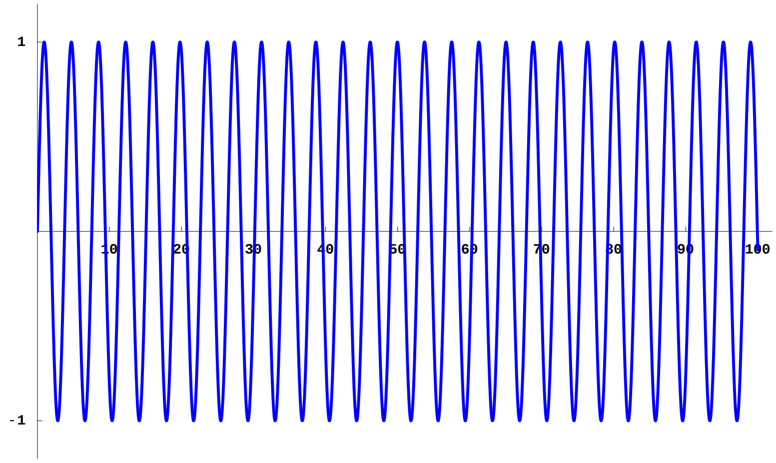
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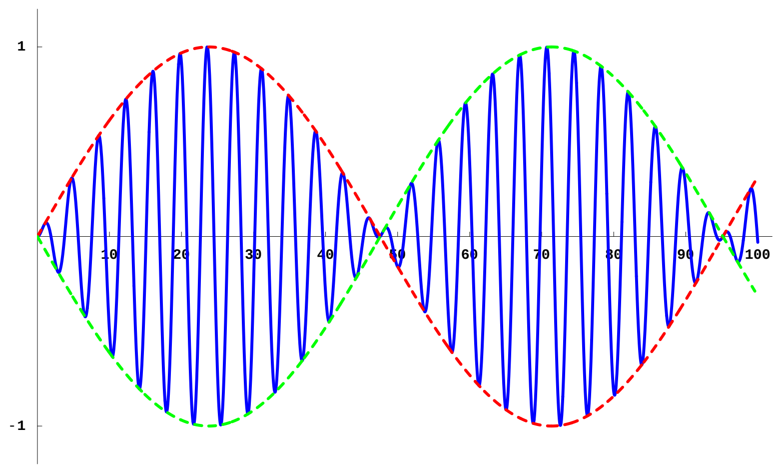
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Let's return to the solution to

$$\frac{d^2y}{dt^2} + 3y = \cos\omega t$$

that satisfies the initial condition $(y(0), y'(0)) = (0, 0)$. If $\omega \neq \pm\sqrt{3}$, the solution is

$$y(t) = \frac{1}{3-\omega^2}(\cos\omega t - \cos\sqrt{3}\,t).$$

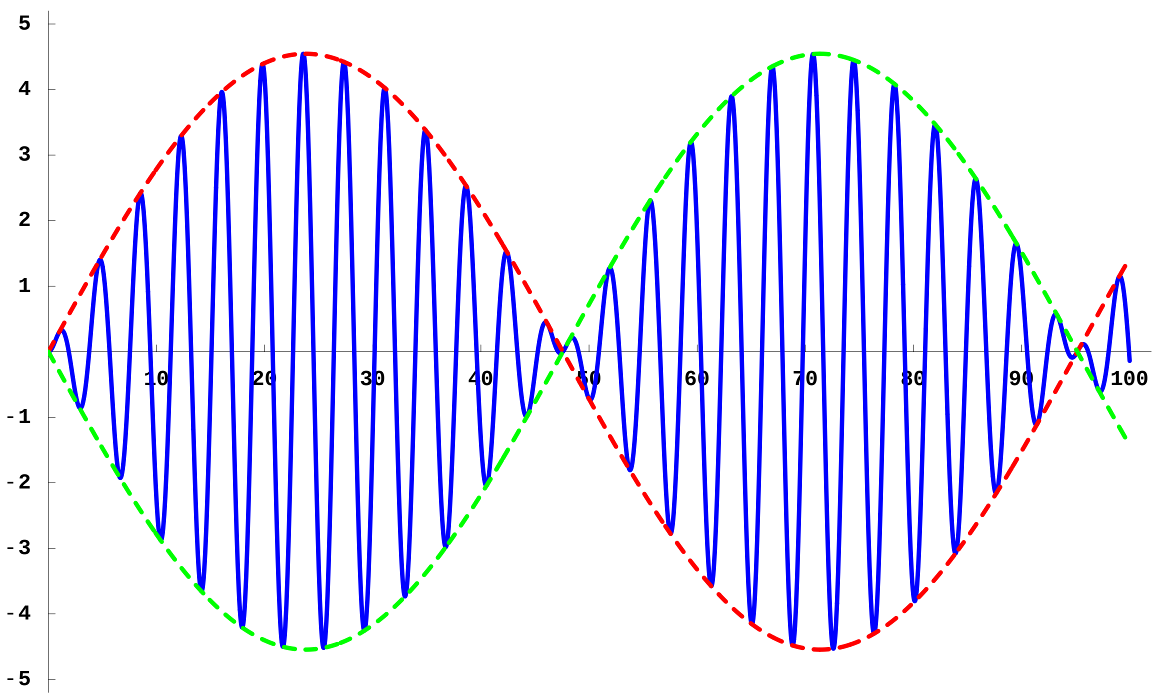
Applying the trig identity, we obtain

$$y(t) = \frac{-2}{3-\omega^2}\,(\sin\alpha t)\,(\sin\beta t)$$

where

$$\alpha = \frac{\omega+\sqrt{3}}{2} \quad \text{and} \quad \beta = \frac{\omega-\sqrt{3}}{2}.$$

Here is the graph of this solution in the case where $\omega = 1.6$.



What happens if $\omega = \sqrt{3}$?

Example.

$$\frac{d^2y}{dt^2} + 3y = \cos \sqrt{3}\, t$$

The complexified equation is

$$\frac{d^2y}{dt^2} + 3y = e^{i\sqrt{3}\, t}.$$

What guess should we use?

Here is the graph of $y_p(t)$.