1. (19 points) Consider the linear system $= A$

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \mathbf{Y}.$$

(a) Determine the type (sink, saddle, source, ...) of the equilibrium point at the origin and find *all* straight-line solutions. Make sure that you show the computations that justify your answers.

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{pmatrix} = (\lambda-1)(\lambda+1) - 3$$
$$= \lambda^2 - 4$$

Eigenvalues: $\lambda = \pm 2$ saddle

$\lambda = 2$ eigenvectors: $\begin{cases} x + 3y = 2x \\ x - y = 2y \end{cases} \Rightarrow x = 3y$

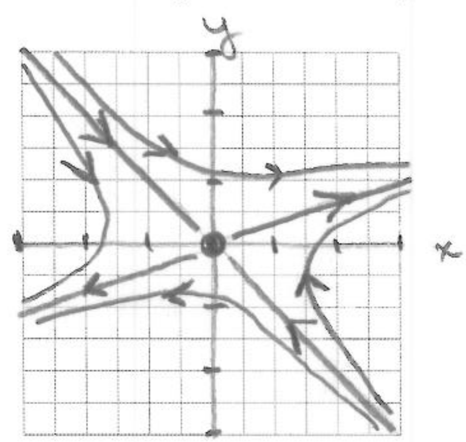
$\lambda = -2$ eigenvectors: $\begin{cases} x + 3y = -2x \\ x - y = -2y \end{cases} \Rightarrow y = -x$

SL solutions:

$\mathbf{Y}(t) = k_1 e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{Y}(t) = k_2 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

where k_1 and k_2 are nonzero constants.

(b) Sketch the phase portrait of this system over the square $-3 \le x \le 3$ and $-3 \le y \le 3$.



2. (20 points) Let $h(y) = y$ if $y > 0$ and $h(y) = 0$ if $y \leq 0$. Consider the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y + 2h(y) = 1, \quad (y_0, v_0) = (-0.1, 0).$$

(a) Reduce this second-order equation to a first-order system and use Euler's method with 5 steps to calculate an approximate solution over the interval $0 \leq t \leq 1$. Enter the results in the table provided. Make sure that you show enough calculations so that the grader can understand how you obtained your answer. You may use a calculator and do all calculations to two decimal places if you wish.

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = 1 - 3y - 2h(y) - 2v$$

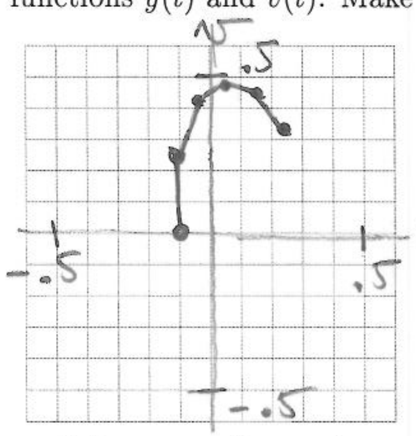
Euler: $\Delta t = \frac{1}{5} = .2$

k	y_k	v_k
0	-0.1	0
1	-0.1	.26
2	-0.05	.42
3	.03	.48
4	.13	.46
5	.22	.35

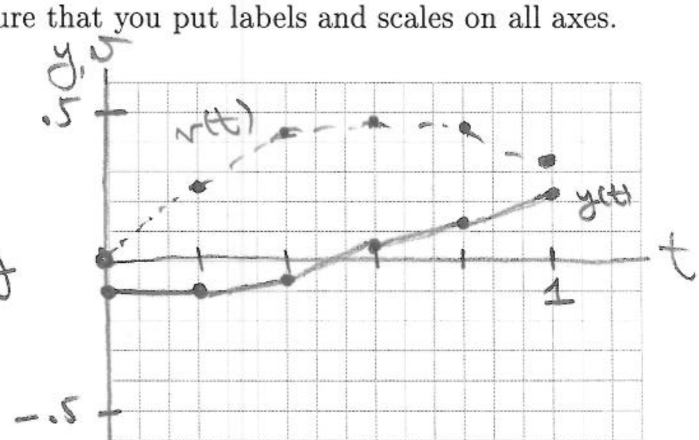
$$y_{k+1} = y_k + (\Delta t)v_k$$

$$v_{k+1} = v_k + (\Delta t)(1 - 3y_k - 2h(y_k) - 2v_k)$$

(b) On the left-hand grid, sketch the approximate solution curve in the yv-phase plane, and on the right-hand grid, sketch the graphs of the approximations to the functions $y(t)$ and $v(t)$. Make sure that you put labels and scales on all axes.



solution curve in yv-plane



$y(t)$- and $v(t)$-graphs

3. (18 points) Short answer questions: The answers to these questions need only consist of one or two sentences, but you do need to show enough work so that we can tell that you are not just guessing. Partial credit will be awarded only in exceptional situations.

(a) How many equilibrium points does the system $dx/dt = x(x-y)$ and $dy/dt = (x^2-4)(y^2-9)$ have? What are they?

$\frac{dx}{dt} = 0 \iff x = 0$ or $x = y$

$\frac{dy}{dt} = 0 \iff x^2 = 4$ or $y^2 = 9$

eq. points: $(0,3)$, $(0,-3)$, $(2,2)$, $(-2,-2)$
$(3,3)$, $(-3,-3)$ six

(b) Find the matrix $\mathbf{A}$ for which the function $\mathbf{Y}(t) = (3\cos 2t, \sin 2t)$ is a solution to $d\mathbf{Y}/dt = \mathbf{AY}$.

$x(t) = 3\cos 2t \Rightarrow dx/dt = -6\sin 2t = -6y(t)$

$y(t) = \sin 2t \Rightarrow dy/dt = 2\cos 2t = \frac{2}{3}x(t)$

$$A = \begin{pmatrix} 0 & -6 \\ \frac{2}{3} & 0 \end{pmatrix}$$

(c) Is the function $(x(t), y(t)) = (e^{-6t}, 2e^{-3t})$ a solution to the system $dx/dt = 2x - 2y^2$ and $dy/dt = -3y$? Why?

$\frac{dx}{dt} = -6e^{-6t}$ and $2x - 2y^2 = 2e^{-6t} - 8e^{-6t}$
$= -6e^{-6t}$ ✓

$\frac{dy}{dt} = -6e^{-3t}$ and $-3y = -3(2e^{-3t})$
$= -6e^{-3t}$ ✓

This is a solution.

4. (20 points) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}.$$

Its eigenvalues are $\lambda = 2 \pm 3i$. Compute the general solution to $d\mathbf{Y}/dt = \mathbf{AY}$.

Eigenvectors for $\lambda = 2+3i$:

$$\begin{cases} x_0 - 2y_0 = (2+3i)x_0 \\ 5x_0 + 3y_0 = (2+3i)y_0 \end{cases} \Longrightarrow -2y_0 = (1+3i)x_0$$

One eigenvector is $\mathbf{Y}_0 = \begin{pmatrix} -2 \\ 1+3i \end{pmatrix}$.

$$\mathbf{Y}_c(t) = e^{(2+3i)t}\begin{pmatrix} -2 \\ 1+3i \end{pmatrix}.$$

$$= e^{2t}(\cos 3t + i \sin 3t)\begin{pmatrix} -2 \\ 1+3i \end{pmatrix}$$

$$\operatorname{Re} \mathbf{Y}_c(t) = e^{2t}\begin{pmatrix} -2\cos 3t \\ \cos 3t - 3\sin 3t \end{pmatrix}$$

$$\operatorname{Im} \mathbf{Y}_c(t) = e^{2t}\begin{pmatrix} -2\sin 3t \\ 3\cos 3t + \sin 3t \end{pmatrix}$$

General solution:

$$\mathbf{Y}(t) = k_1 e^{2t}\begin{pmatrix} -2\cos 3t \\ \cos 3t - 3\sin 3t \end{pmatrix} + k_2 e^{2t}\begin{pmatrix} -2\sin 3t \\ 3\cos 3t + \sin 3t \end{pmatrix}$$

where k_1 and k_2 are arbitrary constants

5. (20 points) Solve the initial-value problem

$$\frac{dx}{dt} = x + 2y + 1 \qquad \frac{dy}{dt} = 3y \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

partially decoupled: $y(t) = 2e^{3t}$

$\Rightarrow \frac{dx}{dt} = x + 4e^{3t} + 1$ linear non homogeneous

Use $\mu = e^{\int -1\,dt} = e^{-t} \Rightarrow$

$(e^{-t})\left(\frac{dx}{dt}\right) - (e^{-t})x = 4e^{2t} + e^{-t}$

$\frac{d}{dt}(e^{-t}x) = 4e^{2t} + e^{-t}$

$\Rightarrow e^{-t}x = 2e^{2t} - e^{-t} + c$

$x = 2e^{3t} - 1 + ce^{t}$

$x(0) = 5 \Rightarrow 5 = 2 - 1 + c \Rightarrow c = 4$

$\Rightarrow x(t) = 2e^{3t} - 1 + 4e^{t}$

$y(t) = 2e^{3t}$