

1. (20 points) Solve the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0, \quad y(0) = 4, \quad y'(0) = -11.$$

char eqn $\lambda^2 + 4\lambda + 13 = 0$

$$\lambda = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i$$

$$y(t) = k_1 e^{-2t} \cos 3t + k_2 e^{-2t} \sin 3t$$

$$y(0) = 4 \Rightarrow k_1 = 4$$

$$y(t) = 4e^{-2t} \cos 3t + k_2 e^{-2t} \sin 3t$$

$$y'(t) = -8e^{-2t} \cos 3t - 12e^{-2t} \sin 3t$$

$$-2k_2 e^{-2t} \sin 3t + 3k_2 e^{-2t} \cos 3t$$

$$y'(0) = -8 + 3k_2 \stackrel{?}{=} -11$$

$$\Rightarrow 3k_2 = -3 \Rightarrow k_2 = -1$$

$$y(t) = 4e^{-2t} \cos 3t - e^{-2t} \sin 3t$$

2. (24 points) Consider the system

$$\begin{aligned}\frac{dx}{dt} &= 10 - x^2 - y^2 \\ \frac{dy}{dt} &= 3x - y.\end{aligned}$$

(a) Determine all equilibrium points of this system.

$$\left\{ \begin{array}{l} \frac{dx}{dt} = 0 \Rightarrow x^2 + y^2 = 10 \Rightarrow x^2 + 9x^2 = 10 \\ \frac{dy}{dt} = 0 \Rightarrow y = 3x \end{array} \right. \begin{array}{l} x^2 = 1 \\ x = \pm 1 \end{array}$$

Two equilibrium points:

$$(1, 3) \text{ and } (-1, -3).$$

(b) Identify the types of the equilibrium points that you found in Part a. In other words, determine if they are sinks, saddles, sources, Make sure that you indicate how you derived your answer.

Jacobian matrix $J(x, y) = \begin{pmatrix} -2x & -2y \\ 3 & -1 \end{pmatrix}$

$$J(1, 3) = \begin{pmatrix} -2 & -6 \\ 3 & -1 \end{pmatrix} \Rightarrow \lambda^2 + 3\lambda + 20 = 0$$

$$\lambda = \frac{-3 \pm \sqrt{9 - 80}}{2}$$

$$J(-1, -3) = \begin{pmatrix} 2 & 6 \\ 3 & -1 \end{pmatrix}$$

spiral sink

$$\begin{aligned}\lambda^2 - \lambda - 20 \\ (\lambda - 5)(\lambda + 4)\end{aligned}$$

$$\lambda = 5, -4 \quad \text{saddle}$$

3. (25 points)

(a) Calculate $\mathcal{L}^{-1} \left[\frac{1}{s^2 - 6s + 8} \right]$.

$$\begin{aligned}\frac{1}{s^2 - 6s + 8} &= \frac{1}{(s-2)(s-4)} = \frac{A}{s-2} + \frac{B}{s-4} \\ &= \frac{(A+B)s - (4A+2B)}{(s-2)(s-4)}\end{aligned}$$

$$\Rightarrow A+B=0 \text{ and } 4A+2B=-1 \Rightarrow A=-\frac{1}{2}, B=\frac{1}{2}$$

$$\mathcal{L}^{-1} \left[\frac{1}{s^2 - 6s + 8} \right] = \frac{1}{2} e^{4t} - \frac{1}{2} e^{2t}.$$

(b) Compute $\mathcal{L}[u_4(t)t]$ directly from the definition of the Laplace transform.

$$\begin{aligned}\mathcal{L}[u_4(t)t] &= \int_0^\infty u_4(t)t e^{-st} dt \\ &= \int_4^\infty t e^{-st} dt\end{aligned}$$

Integrate $\int t e^{-st} dt$ by parts:

$$u=t \quad dv = e^{-st} dt$$

$$du = dt \quad v = -\frac{1}{s} e^{-st}$$

$$\begin{aligned}\int t e^{-st} dt &= -\frac{1}{s} t e^{-st} + \frac{1}{s} \int e^{-st} dt \\ &= -\frac{1}{s} t e^{-st} - \frac{1}{s^2} e^{-st} + C\end{aligned}$$

3(b) continued

$$\begin{aligned}
 \mathcal{L}[u_4(t)t] &= \int_4^\infty t e^{-st} dt \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_4^b \\
 &= \lim_{b \rightarrow \infty} \left[\frac{te^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_b^4 \\
 &= \frac{4e^{-4s}}{s} + \frac{e^{-4s}}{s^2} - \\
 &\quad \lim_{b \rightarrow \infty} \left[\frac{be^{-sb}}{s} + \frac{e^{-sb}}{s^2} \right].
 \end{aligned}$$

If $s > 0$, then

$$\lim_{b \rightarrow \infty} b e^{-sb} = 0 \quad \text{and}$$

$$\lim_{b \rightarrow \infty} e^{-sb} = 0.$$

$$\begin{aligned}
 \Rightarrow \mathcal{L}[u_4(t)t] &= \frac{4e^{-4s}}{s} + \frac{e^{-4s}}{s^2} \\
 &= \left(\frac{4}{s} + \frac{1}{s^2} \right) e^{-4s}.
 \end{aligned}$$

See computations on subsequent pages

4. (28 points) On the next page, there are eight second-order linear equations and four graphs of solutions. Match each graph with its corresponding equation. Provide a brief justification for your choice. You will not receive any credit unless you provide a valid justification.

- (a) The equation for graph A is 6. My reason for choosing this answer is:

Beats with slow oscillations
having a period slightly less
than 30. #6 is the only possibility.

- (b) The equation for graph B is 2. My reason for choosing this answer is:

Periodic solution that oscillates
around $y=2 \Rightarrow \#1$ or $\#2$.

Period of oscillation is slightly
less than 3.

- (c) The equation for graph C is 7. My reason for choosing this answer is:

Periodic solution that oscillates
around $y=0 \Rightarrow \#7$ or $\#8$

Amplitude of steady state in
 $\#7$ is approx 2.4

- (d) The equation for graph D is 5. My reason for choosing this answer is:

This graph corresponds either to
resonance or to a beat whose
slow oscillation has period at
least 80. We do not have any
beats with this large a period
 \Rightarrow resonance ($\#5$).

Computations to support justifications
in Problem #4:

1) $\frac{d^2y}{dt^2} + 3y = 6$ has constant function
 $y_p(t) = 2$ as a solution.

Undamped \Rightarrow solutions are periodic
with oscillations centered around $y = 2$
and period $2\pi/\sqrt{3} \approx 3.6$

2) $\frac{d^2y}{dt^2} + 5y = 10$ has constant function
 $y_p(t) = 2$ as a solution

Undamped \Rightarrow solutions are periodic
with oscillations centered around $y = 2$
and period $2\pi/\sqrt{5} \approx 2.8$.

3) $\frac{d^2y}{dt^2} + 4y = -8$ has constant function
 $y_p(t) = -2$ as a
solution.

Undamped \Rightarrow solutions are periodic
with oscillations centered around
 $y = -2$.

#4 - supporting computations:

4) $\frac{d^2y}{dt^2} + 3y = \cos 9t$ (beats)

natural frequency = $\sqrt{3}$

forcing frequency = 9

half difference = 3.6

\Rightarrow slow oscillations have period 1.7

5) $\frac{d^2y}{dt^2} + 9y = \cos 3t$

natural frequency = 3

forcing frequency = 3

\Rightarrow resonance

6) $\frac{d^2y}{dt^2} + 12y = 6 \cos 3t$ (beats)

natural frequency = $\sqrt{12}$

forcing frequency = 3

half difference = .23

\Rightarrow slow oscillations have period 27..

#4 - supporting computations:

7 and 8 have same left-hand side
only difference:

$$\text{forcing for } \#7 = 10 \cos 2t$$

$$\text{forcing for } \#8 = \cos 2t$$

Calculate amplitude of steady-state
solution for #8 first:

complexify and guess $y_c(t) = ae^{(2i)t}$

We have a solution if

$$a(-4 + 4i + 3) \stackrel{?}{=} 1$$

$$\Rightarrow a = \frac{1}{-1+4i}$$

amplitude of steady state solution

$$\text{is } |a| = \frac{1}{\sqrt{17}} \approx 0.24.$$

The forcing for #7 larger by a
factor of 10. \Rightarrow

amplitude of steady state ≈ 2.4

4. (continued) Answer this question on the previous page.

The 8 second-order linear equations:

$$1. \frac{d^2y}{dt^2} + 3y = 6$$

$$2. \frac{d^2y}{dt^2} + 5y = 10$$

$$3. \frac{d^2y}{dt^2} + 4y = -8$$

$$4. \frac{d^2y}{dt^2} + 3y = \cos 9t$$

$$5. \frac{d^2y}{dt^2} + 9y = \cos 3t$$

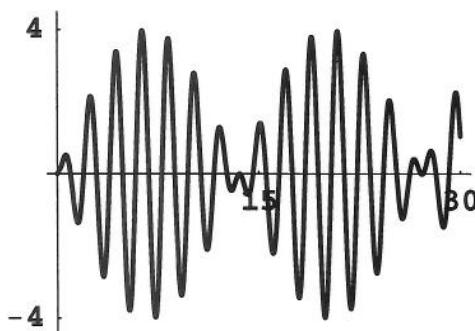
$$6. \frac{d^2y}{dt^2} + 12y = 6 \cos 3t$$

$$7. \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 10 \cos 2t$$

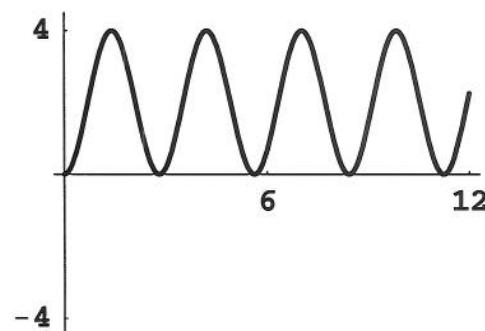
$$8. \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = \cos 2t$$

The four graphs of solutions:

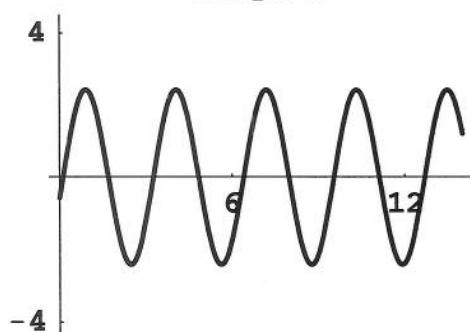
Graph A



Graph B



Graph C



Graph D

