

More on the classification of the damped harmonic oscillator

On Friday we were applying what we learned about homogeneous second-order equations to the (damped) harmonic oscillator

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = 0.$$

In this case, we are assuming that the parameters  $m$  and  $k$  are positive and that  $b \geq 0$ . The characteristic equation  $m\lambda^2 + b\lambda + k = 0$  has eigenvalues

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are three cases based on the value of the discriminant  $b^2 - 4mk$ .

1.  $b^2 - 4mk < 0$ : In this case, the eigenvalues are complex and can be written as

$$\lambda = \left(-\frac{b}{2m}\right) \pm \left(\frac{\sqrt{4mk - b^2}}{2m}\right) i.$$

The real part determines the exponential decay rate for solutions, and the imaginary part determines the natural “period” of the solutions.

There are two subcases:

- (a)  $b = 0$ : all solutions are periodic. This is the **undamped** case.
- (b)  $b \neq 0$ : solutions oscillate with a constant frequency, but they decay at an exponential rate. This is the **underdamped** case.

2.  $b^2 - 4mk = 0$ : The eigenvalue

$$\lambda = -\frac{b}{2m}$$

is repeated. This is the **critically damped** case. In this case, solutions approach zero as rapidly as possible.

3.  $b^2 - 4mk > 0$ :

(Additional blank space on the top of the next page.)

**Example.** Consider the one-parameter family of equations

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + y = 0.$$

We can see the progression from underdamped to critically damped to overdamped with a Quicktime animation that I have posted on the web site.

The trace-determinant plane

There is a nice geometric object called the trace-determinant plane that organizes the various types of  $2 \times 2$  linear systems.

Consider the  $2 \times 2$  matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let's calculate the characteristic polynomial of  $\mathbf{A}$ :

Conclusion: The eigenvalues of any  $2 \times 2$  matrix are determined by the trace and the determinant of  $\mathbf{A}$ . We have

$$\lambda = \frac{(\operatorname{tr} \mathbf{A}) \pm \sqrt{(\operatorname{tr} \mathbf{A})^2 - 4(\det \mathbf{A})}}{2}.$$

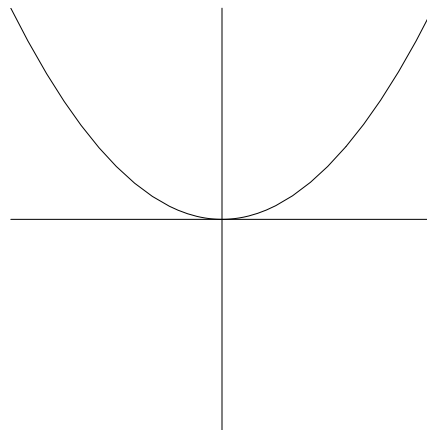
## Summary of Phase Portraits

Assume  $\det \mathbf{A} \neq 0$ . Then zero is not an eigenvalue of  $\mathbf{A}$ .

1. Real and distinct eigenvalues
  - (a) sink
  - (b) saddle
  - (c) source
2. Complex eigenvalues
  - (a) spiral sink
  - (b) center
  - (c) spiral source
3. Real and repeated eigenvalues
  - (a) sink with one eigenline in the phase portrait
  - (b) source with one eigenline in the phase portrait
  - (c) sink where every solution is a straight-line solution
  - (d) source where every solution is a straight-line solution

What if  $\det \mathbf{A} = 0$ ?

We can organize these different types using a plane with unusual coordinate axes.



You can turn on the trace-determinant plane in the `LinearPhasePortraits` tool.

**Example.** Consider the one-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 1 \\ 0 & d \end{pmatrix} \mathbf{Y}.$$

