Existence and Uniqueness Theory

First we consider three examples to illustrate the idea of the domain of a differential equation:

**Example 1.** $\frac{dy}{dt} = y^3 + t^2$

**Example 2.** $\frac{dy}{dt} = y^2$

**Example 3.** $\frac{dy}{dt} = \frac{y}{t}$

We start our discussion of the theory with the Existence Theorem:

**Existence Theorem** Suppose $f(t, y)$ is a continuous function in a rectangle of the form

$$\{(t, y) \mid a < t < b, \ c < y < d\}$$

in the $ty$-plane. If $(t_0, y_0)$ is a point in this rectangle, then there exists an $\epsilon > 0$ and a function $y(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ that solves the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. \quad \blacksquare$$
What’s the significance of the $\epsilon$ in the Existence Theorem?

**Example.** $\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0$

What does the Existence Theorem tell us about the initial-value problem

$$\frac{dy}{dt} = y^3 + t^2, \quad y(0) = 0?$$
The other main theoretical result in differential equations is the Uniqueness Theorem.

**Uniqueness Theorem**  Suppose $f(t, y)$ and $\partial f/\partial y$ are continuous functions in a rectangle of the form
\[
\{(t, y) \mid a < t < b, c < y < d\}
\]
in the $ty$-plane. If $(t_0, y_0)$ is a point in this rectangle and if $y_1(t)$ and $y_2(t)$ are two functions that solve the initial-value problem
\[
\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0
\]
for all $t$ in the interval $t_0 - \epsilon < t < t_0 + \epsilon$ (where $\epsilon$ is some positive number), then
\[
y_1(t) = y_2(t)
\]
for $t_0 - \epsilon < t < t_0 + \epsilon$. That is, the solution to the initial-value problem is *unique.* \[\blacksquare\]

Here’s an example that lacks uniqueness:

**Example.** $\frac{dy}{dt} = \sqrt{y}$

(More blank space and the slope field for $dy/dt = \sqrt{y}$ on the top of the next page.)
Bogus Example. The example
\[ \frac{dy}{dt} = \frac{y}{t} + t \cos t \]
in FirstOrderSystems seems to violate the Uniqueness Theorem, but in fact it does not. Why?