More on the guessing technique for the damped harmonic oscillator

**Example.** Last class we applied a guessing technique to produce two solutions for the harmonic oscillator equation

\[
\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0.
\]

Using the characteristic equation, we obtained

\[y_1(t) = e^{-2t} \quad \text{and} \quad y_2(t) = e^{-t}.
\]

The corresponding velocity functions are

\[v_1(t) = -2e^{-2t} \quad \text{and} \quad v_2(t) = -e^{-t}.
\]

In vector form, these solutions are written as

\[Y_1(t) = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}
\]

and

\[Y_2(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

Let’s plot these solutions with HPGSystemSolver. What are the corresponding solution curves and component graphs?
Euler’s method for a system

We can use the vector field for a system to produce numerical approximations for the solutions.

**Example.** Consider the initial-value problem

\[
\frac{dx}{dt} = -y \quad \frac{dy}{dt} = x - y \quad (x_0, y_0) = (2, 0).
\]

The `EulersMethodForSystems` tool demonstrates the method. We pick a large step size \( \Delta t = 0.5 \) so that we can see the method in action.

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Now let’s derive the general equations for Euler’s method for an autonomous initial-value problem of the form
\[
\frac{dx}{dt} = f(x, y) \quad (x(t_0), y(t_0)) = (x_0, y_0).
\]
\[
\frac{dy}{dt} = g(x, y)
\]
Euler’s method for systems is just as easy to program as Euler’s method for equations. Once again here’s how we can program it with a spreadsheet.

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There are two spreadsheets posted on the course web site—one for the example above and one for the following example.

**Example.** Consider the predator-prey system

\[
\frac{dR}{dt} = R - 0.2RF
\]

\[
\frac{dF}{dt} = -0.3F + 0.1RF
\]

along with the initial condition \((R_0, F_0) = (1, 2)\). Using the spreadsheet on the web site, we see that Euler’s method has trouble approximating periodic solutions.

**HPGSystemSolver** uses a more sophisticated fixed-step-size algorithm called the Runge-Kutta method. It usually works better than Euler’s method, but there are equations for which any fixed-step-size algorithm is not appropriate.
Existence and Uniqueness Theory for Systems

There is an existence and uniqueness theorem for systems just like the theorem for equations.

**Existence and Uniqueness Theorem.** Let

\[
\frac{dY}{dt} = F(t, Y)
\]

be a system of differential equations. Suppose that \( t_0 \) is an initial time and \( Y_0 \) is an initial value. Suppose also that the function \( F \) is continuously differentiable. Then there is an \( \epsilon > 0 \) and a function \( Y(t) \) defined for \( t_0 - \epsilon < t < t_0 + \epsilon \) such that

\[
\frac{dY}{dt} = F(t, Y(t)) \quad \text{and} \quad Y(t_0) = Y_0.
\]

In other words, \( Y(t) \) satisfies the initial-value problem. Moreover, for \( t \) in this interval, this solution is unique.

There is an important consequence of the Uniqueness Theorem for autonomous systems: Consider the metaphor of the parking lot.

Given the autonomous system

\[
\frac{dY}{dt} = F(Y).
\]

Let \( Y_0 \) be an initial condition such that \( Y_1(t) \) is a solution that satisfies \( Y(t_1) = Y_0 \) and \( Y_2(t) \) is another solution that satisfies \( Y(t_2) = Y_0 \). Then

\[
Y_2(t) = Y_1(t - (t_2 - t_1)).
\]
Example. Consider the second-order equation

\[
\frac{d^2y}{dt^2} + y = 0,
\]

which is equivalent to the system

\[
\begin{align*}
\frac{dy}{dt} &= v \\
\frac{dv}{dt} &= -y.
\end{align*}
\]

Note that

\[
Y_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \text{and} \quad Y_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}
\]

are both solutions to the system. How are \(Y_1(t)\) and \(Y_2(t)\) related?

There is an animation on the web site that illustrates this phenomenon.

Here is an informal restatement of this consequence of uniqueness:

For an autonomous system, if two solution curves in the phase plane touch, then they are identical.