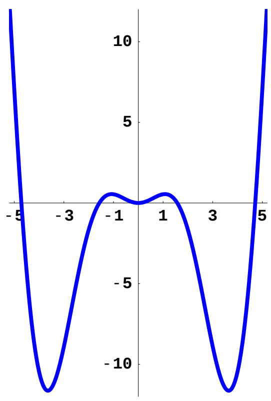
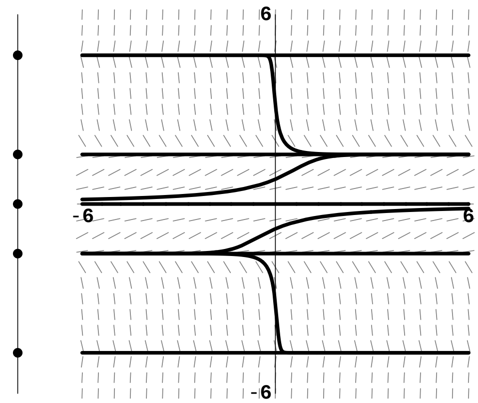
A little more about phase lines

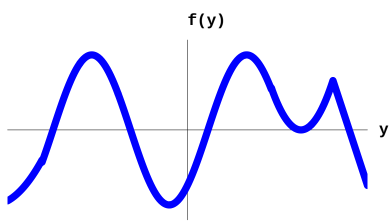
At the end of last class we discussed the following example.

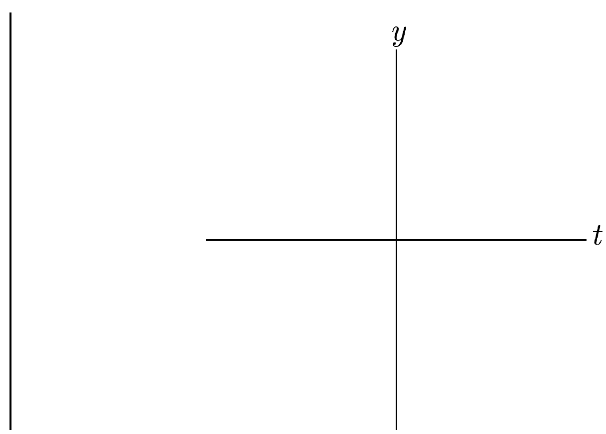
Example 1. $\dfrac{dy}{dt} = y^2 \cos y$





Example 2. $\dfrac{dy}{dt} = f(y)$ where $f(y)$ is given by the graph





Linear differential equations

A first-order differential equation

$$\frac{dy}{dt} = f(t, y)$$

with independent variable t and dependent variable y is **linear** if it can be written as

$$\frac{dy}{dt} = a(t)y + b(t).$$

In other words, the *dependent* variable only appears linearly in the equation.

Linear differential equations:

$$\frac{dy}{dt} = 5y$$

$$\frac{dy}{dt} = (\cos t)y$$

$$\frac{dy}{dt} = y - t^2$$

Nonlinear differential equations:

$$\frac{dy}{dt} = t \cos y$$

$$\frac{dy}{dt} = y^2 - t$$

The linear differential equation

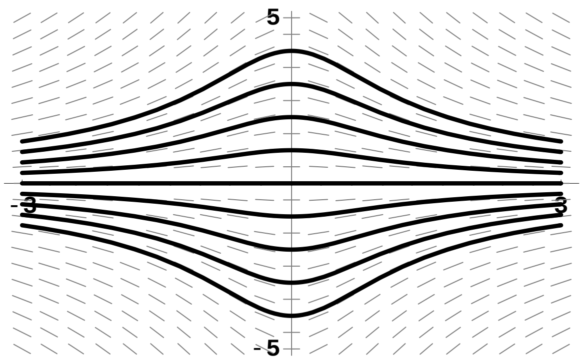
$$\frac{dy}{dt} = a(t)y + b(t)$$

is **homogeneous** if $b(t) = 0$ for all t. Otherwise, it is **nonhomogeneous**. (Some people use the term inhomogeneous.)

Where have we seen homogeneous linear differential equations before?

Example. $\dfrac{dy}{dt} = \dfrac{-ty}{1+t^2}$

(Graphs and slope field on top of next page.)



Linearity Principles

Why are linear equations so much more amenable to analytic techniques than nonlinear equations? The reason is that their solutions satisfy important linearity principles.

Let's begin with homogeneous linear equations:

Linearity Principle. If $y_h(t)$ is a solution of a homogeneous linear differential equation

$$\frac{dy}{dt} = a(t)y,$$

then any *constant* multiple $y_k(t) = ky_h(t)$ of $y_h(t)$ is also a solution. In other words, given a constant $k \neq 1$ and a solution $y_h(t)$, we obtain another solution by multiplying $y_h(t)$ by k.