Linear systems

Last class we started to discuss linear systems, that is, the ones that can be written as

$$\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy\end{aligned} \qquad \text{or as} \qquad \frac{d\mathbf{Y}}{dt} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{where} \qquad \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The numbers a, b, c, and d are constants. These constants are also referred to as the coefficients or as the parameters of the system.

Last class we also reviewed two examples that we had discussed previously.

Example 1. We have already calculated the general solution to the partially decoupled system

$$\begin{aligned}\frac{dx}{dt} &= 2y - x \\ \frac{dy}{dt} &= y.\end{aligned}$$

Written in vector notation, the general solution is

$$\mathbf{Y}(t) = e^t \begin{pmatrix} y_0 \\ y_0 \end{pmatrix} + e^{-t} \begin{pmatrix} x_0 - y_0 \\ 0 \end{pmatrix}.$$

Example 2. For the damped harmonic oscillator

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0,$$

we used a guessing technique to find the two (scalar) solutions $y_1(t) = e^{-t}$ and $y_2(t) = e^{-2t}$. As usual, this second-order equation can be converted to a first-order system where

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -2y - 3v,\end{aligned}$$

and the two scalar solutions yield two vector solutions

$$\mathbf{Y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Today we will learn that once we have these two solutions in Example 2 we know the general solution.

Given a linear system $\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$, how do we calculate the vector in the vector field at any given point $\mathbf{Y}_0$?

How do we calculate the equilibrium points of $\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$?

Example. Let $\mathbf{A}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

Example. Let $\mathbf{A}_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.

Theorem. The origin is always an equilibrium point of a linear system. It is the only equilibrium point if and only if $\det \mathbf{A} \neq 0$.

The Linearity Principle

Let's return to Example 1. For practice, we'll use vector notation this time:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

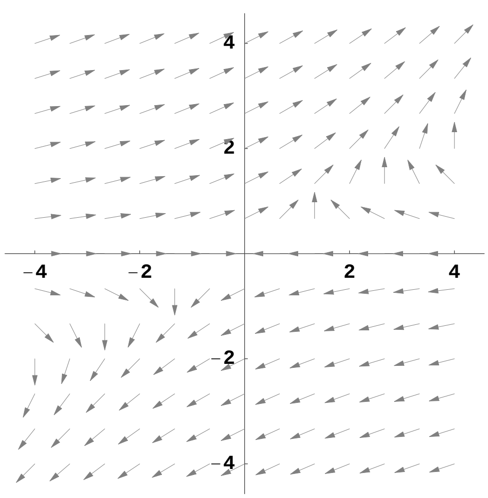
Also consider three different initial conditions

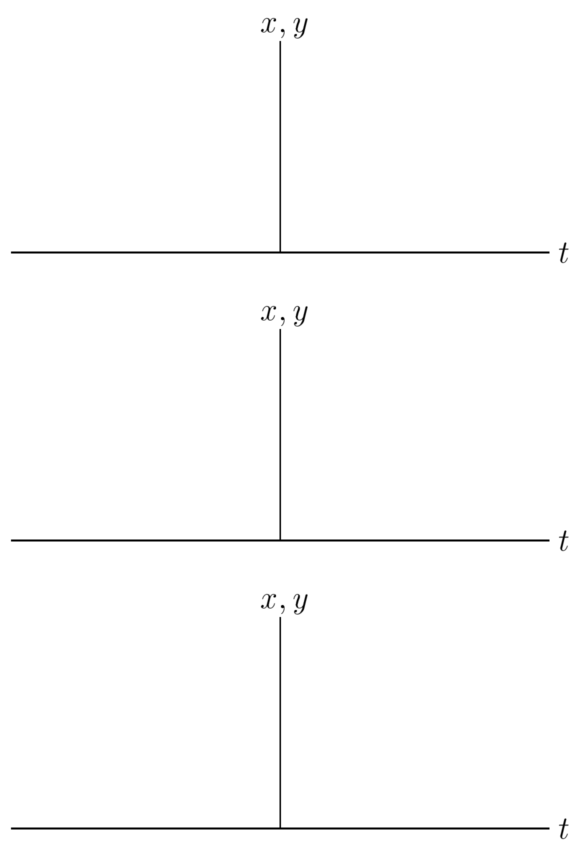
$$\mathbf{Y}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{Y}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{Y}_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

They correspond to the three solutions

$$\mathbf{Y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_2(t) = e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{Y}_3(t) = \begin{pmatrix} e^t + e^{-t} \\ e^t \end{pmatrix}.$$

Let's see what happens when we graph these solutions.





How are these three solutions related?

Linearity Principle Suppose

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

is a linear system of differential equations.

1. If $\mathbf{Y}(t)$ is a solution of this system and k is any constant, then $k\mathbf{Y}(t)$ is also a solution.
2. If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are two solutions of this system, then $\mathbf{Y}_1(t)+\mathbf{Y}_2(t)$ is also a solution.

This principle gives us a more general way to find solutions of linear systems. To see how this approach works, let's consider Example 1 again along with the two solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.

Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

and the two solutions

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}.$$

Any linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ is also a solution to the system.

Example. Solve

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

For an arbitrary linear system $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$, how many solutions do we need to solve every initial-value problem?