Linear systems—a brief review

A linear system (with constant coefficients) can be written as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y},$$

where $\mathbf{A}$ is a square matrix of constants (the coefficients). For us, $\mathbf{A}$ will be a 2×2 matrix. Using the Linearity Principle, we can produce many solutions from just a few:

If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, then

$$k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$$

is a solution for any choice of constants k_1 and k_2.

"Straight-line" Solutions. Suppose that

$$\mathbf{A}\mathbf{Y}_0 = \lambda \mathbf{Y}_0$$

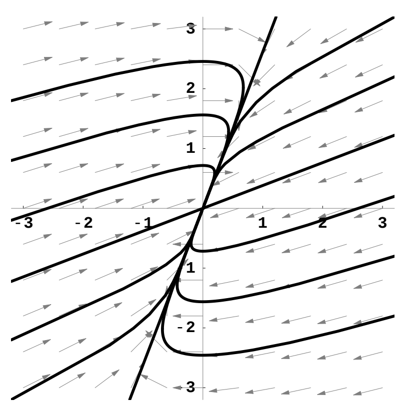
for some nonzero vector $\mathbf{Y}_0$ and some scalar λ. Then the function

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$$

is a solution to the linear differential equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}.$$

Real and distinct (nonzero) eigenvalues $\lambda_1 < \lambda_2$



sink ($\lambda_1 < \lambda_2 < 0$)

More on the example from last class

Example. Once again consider

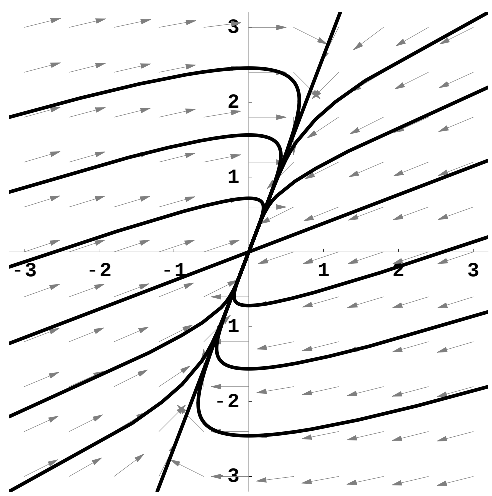
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$

For this example, the eigenvalues are

$$\lambda = \frac{-3 \pm \sqrt{5}}{2}.$$

Both are negative.

The slope of the eigenline that corresponds to the "fast" eigenvalue $\lambda_1 = \frac{1}{2}(-3 - \sqrt{5})$ is approximately 0.4, and the slope of the eigenline that corresponds to the "slow" eigenvalue $\lambda_2 = \frac{1}{2}(-3 + \sqrt{5})$ is approximately 2.6.

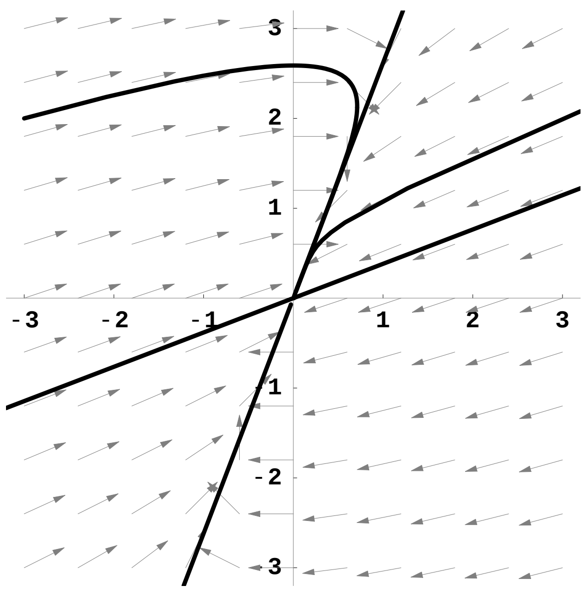


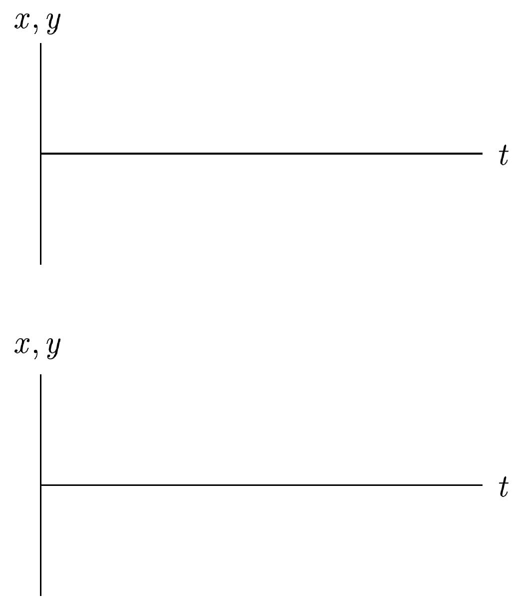
Sketching component graphs

Once we understand the phase portrait, we should also be able to sketch the component graphs without `HPGSystemSolver`.

Let's sketch the $x(t)$- and $y(t)$-graphs that correspond to the initial conditions $(-3, 2)$ and $(3, 2)$ for

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$



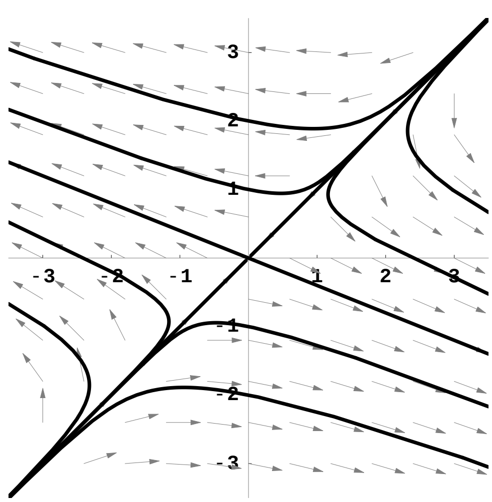


Case 2: $\lambda_1 < 0 < \lambda_2$.

We did this case at the start of class on March 5.

Example. Consider

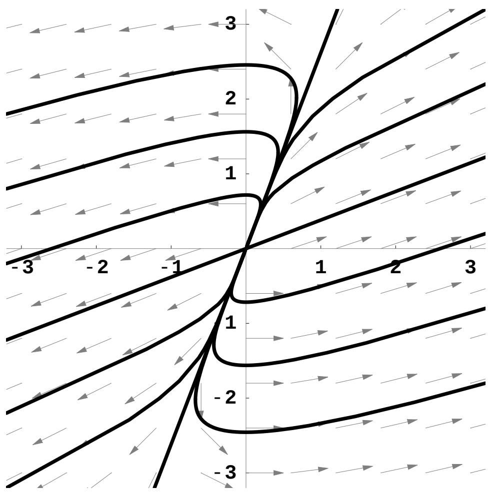
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$



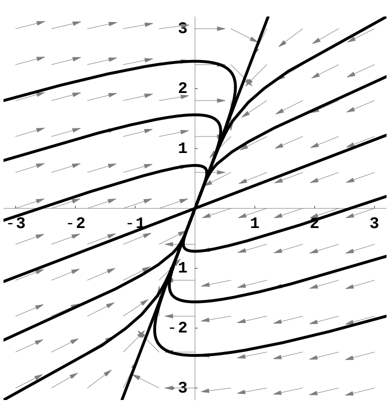
Case 3: $0 < \lambda_1 < \lambda_2$.

Example. Consider

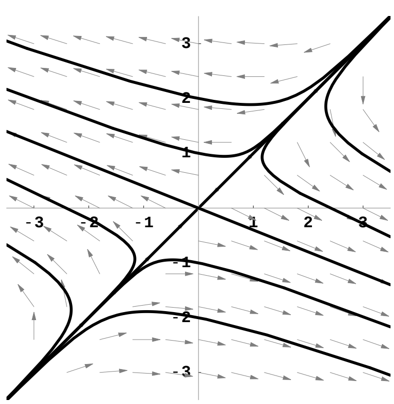
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{Y}.$$



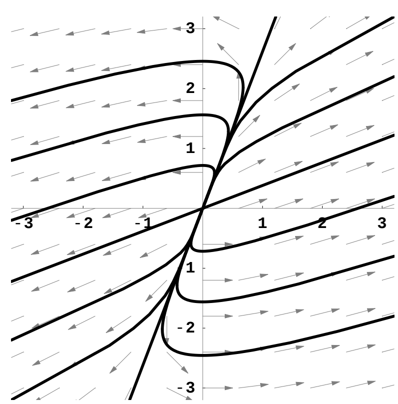
Summary for real and distinct (nonzero) eigenvalues



sink ($\lambda_1 < \lambda_2 < 0$)



saddle ($\lambda_1 < 0 < \lambda_2$)



source ($0 < \lambda_1 < \lambda_2$)

Complex eigenvalues

What happens if the eigenvalues of the system are complex numbers?

Example. Consider
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{Y}.$$

Let's see that happens if we take a look at this system using `MatrixFields` and then we'll compute the eigenstuff for this matrix.

Eigenvalues:

Eigenvectors:

(Lots of blank space on the next page.)

We now have a complex-valued solution of the form

$$\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}.$$

There are lots of questions that come with this formula. First, what does the formula mean? Second, what good is it given that we are interested in real-valued solutions to our linear systems?