Last class we studied an example with complex eigenvalues.

**Example.** Consider $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$.

Its eigenvalues are $\lambda = -2 \pm i$, and by solving the eigenvector equation $\mathbf{AY}_0 = (-2+i)\mathbf{Y}_0$, we derived the "straight-line solution"

$$\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}.$$

There are lots of questions that come with this formula. First, what does the formula mean? Second, what good is it given that we are interested in real-valued solutions to our linear systems?

Once again Euler comes to the rescue: Remember the power series for the exponential function? It is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

Let's use this series where $x = bi$.

We use Euler's formula

$$e^{bi} = \cos b + i \sin b$$

applied to the complex-valued function $e^{(a+bi)t}$.

But why does this help us solve our differential equation?

**Theorem.** Consider $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$, where $\mathbf{A}$ is a matrix with real entries. If $\mathbf{Y}_c(t)$ is a complex-valued solution, then both

$$\mathrm{Re}\mathbf{Y}_c(t) \quad \text{and} \quad \mathrm{Im}\mathbf{Y}_c(t)$$

are real-valued solutions, and they are linearly independent.

Now we can derive the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{Y}$$

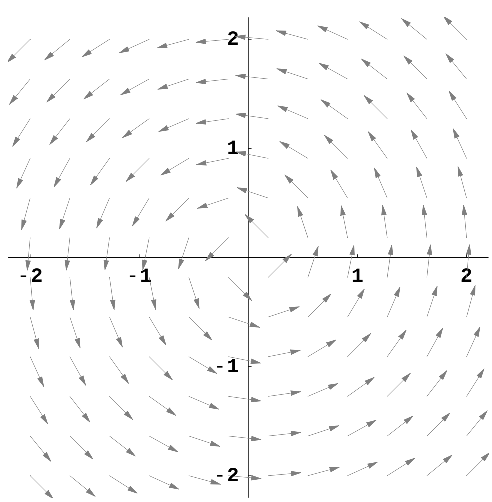
using the complex-valued solution $\mathbf{Y}_c(t) = e^{(-2+i)t} \begin{pmatrix} 2 \\ 1+i \end{pmatrix}$.

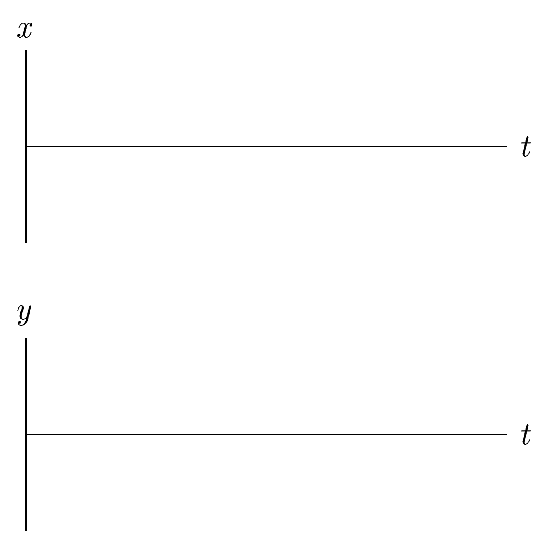
Three examples to illustrate the geometry of complex eigenvalues:

**Example 1.** $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 + 1$, so the eigenvalues are $\lambda = \pm i$. One eigenvector associated to the eigenvalue $\lambda = i$ is

$$\mathbf{Y}_0 = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$



**Example 2.** $\frac{d\mathbf{Y}}{dt} = \mathbf{BY}$ where $\mathbf{B} = \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}$.

The characteristic polynomial of $\mathbf{B}$ is $\lambda^2 + 4$, so the eigenvalues are $\lambda = \pm 2i$. One eigenvector associated to the eigenvalue $\lambda = 2i$ is

$$\mathbf{Y}_0 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}.$$