Repeated eigenvalues

Sometimes the characteristic polynomial has the same real root twice. When this happens, we say that the eigenvalues are “repeated.”

**Example.** \( \frac{dY}{dt} = AY \) where \( A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} \).

The characteristic polynomial of \( A \) is \((\lambda - 3)^2\), so there is only one eigenvalue, \( \lambda = 3 \). Let’s calculate the associated eigenvectors:

But we already know how to solve this system. How?
We obtain the general solution

\[ Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0e^{3t} + 2y_0te^{3t} \\ y_0e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{3t} \begin{pmatrix} 2y_0 \\ 0 \end{pmatrix}. \]

Note that this general solution is \textit{not} written as a linear combination. Every nontrivial solution contains the first term, and most solutions contain both terms.

We use this result to motivate a different technique that we use to solve systems with repeated eigenvalues. We use a guessing technique where we guess a solution of the form

\[ Y(t) = e^{\lambda t}V_0 + te^{\lambda t}V_1. \]

Note that the initial condition for this solution is \( V_0 \).
Fact from linear algebra: If $A$ is a $2 \times 2$ matrix with a repeated eigenvalue $\lambda$ and $V_0$ is any vector, then either

1. $(A - \lambda I)V_0 = 0$ (in other words, $V_0$ is an eigenvector), or

2. the vector $V_1 = (A - \lambda I)V_0$ is an eigenvector of $A$.

Example. $\frac{dY}{dt} = AY$ where

$$A = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}.$$ 

The characteristic polynomial of $A$ is $\lambda^2 + 4\lambda + 4$, so $\lambda = -2$ is a repeated eigenvalue.
What is the long-term behavior of a system with a repeated, negative eigenvalue?

It is interesting to look at this example using two of the tools on the CD. Using LinearPhasePortraits, we can see that this system is on the boundary between spiral sinks and real sinks.

We can also use HPGSystemSolver to plot the phase portrait and a typical pair of $x(t)$- and $y(t)$-graphs.
Unusual case of repeated eigenvalues: There is one type of linear system that has repeated eigenvalues that is different from the examples we have discussed.

**Example.** Consider \( d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y} \) where \( \mathbf{A} \) is the diagonal matrix

\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda \\
\end{pmatrix}.
\]

What are its eigenvalues and eigenvectors?
Finally consider the example

\[
\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \mathbf{Y}.
\]

Its characteristic polynomial is \( \lambda^2 + 3\lambda \). So its eigenvalues are \( \lambda = -3 \) and \( \lambda = 0 \). (If a system has 0 as an eigenvalue, we say that it is degenerate. The matrix \( \mathbf{A} \) of coefficients is singular—see your class notes for March 1 and March 3.)