Second-order, linear equations

We now apply what we have learned about linear systems to solve second-order homogeneous linear equations.

Let's return to the guessing technique for second-order equations that we learned about a month ago (see Section 2.3 in the text and your class notes from February 22 and February 24). In particular, let's see how it relates to what we have done with linear systems recently.

Example. Consider the equation $2\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 4y = 0$.

1. Use a guessing technique to find two nonzero solutions $y_1(t)$ and $y_2(t)$ that are not multiples of each other.

2. Convert this equation to a first-order system and determine the analogous solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.

3. In what way are $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ special solutions?

Let's see how this guessing technique can be used to solve all second-order homogeneous equations.

Consider

$$a\,\frac{d^2y}{dt^2} + b\,\frac{dy}{dt} + cy = 0$$

with its characteristic equation $a\lambda^2 + b\lambda + c = 0$ as well as the corresponding system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{c}{a}\,y - \frac{b}{a}\,v \end{aligned}$$

with its characteristic equation

$$\det \begin{pmatrix} -\lambda & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} - \lambda \end{pmatrix} = 0.$$

Useful observation: If λ is an eigenvalue, the vector $\mathbf{Y}_0 = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$ is *always* an associated eigenvector.

Let's see what that observation tells us about solutions to the second-order equation. There are three cases:

1. Two real, distinct, nonzero eigenvalues λ_1 and λ_2:

2. A complex-conjugate pair of eigenvalues $\lambda = \alpha \pm i\beta$, with $\beta \neq 0$:

3. One nonzero real eigenvalue λ of multiplicity two:

Conclusion: We can determine the general solution of a homogeneous linear second-order equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

immediately from the characteristic equation $a\lambda^2 + b\lambda + c = 0$.

YOU DO NOT NEED TO CALCULATE THE EIGENVECTORS OR EVEN REDUCE TO A FIRST-ORDER SYSTEM if you simply want to produce the general solution of a linear second-order equation.

Example. Let's compute the general solution to $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 0$.

Application: We can apply what we have learned to the (damped) harmonic oscillator

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

In this case, we are assuming that the parameters m and k are positive and that $b \geq 0$. The characteristic equation $m\lambda^2 + b\lambda + k = 0$ has eigenvalues

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are three cases based on the value of the discriminant $b^2 - 4mk$.

1. $b^2 - 4mk < 0$

2. $b^2 - 4mk = 0$

3. $b^2 - 4mk > 0$

We can see the progression from underdamped to critically damped to overdamped with a Quicktime animation I have posted on the web site.