A little more about linear systems/equations

We can apply what we have learned about homogeneous second-order equations to the (damped) harmonic oscillator
$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$
In this case, we are assuming that the parameters m and k are positive and that $b \geq 0$. The characteristic equation $m\lambda^2 + b\lambda + k = 0$ has eigenvalues
$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are three cases based on the value of the discriminant $b^2 - 4mk$.

1. $b^2 - 4mk < 0$:

2. $b^2 - 4mk = 0$:

3. $b^2 - 4mk > 0$:

Example. Consider the one-parameter family of equations

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + y = 0.$$

We can see the progression from underdamped to critically damped to overdamped with a Quicktime animation that I have posted on the web site.

The trace-determinant plane

There is a nice geometric object called the trace-determinant plane that organizes the various types of 2×2 linear systems.

Consider the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let's calculate the characteristic polynomial of $\mathbf{A}$:

Conclusion: The eigenvalues of any 2×2 matrix are determined by the trace and the determinant of $\mathbf{A}$. We have

$$\lambda = \frac{(\mathrm{tr}\mathbf{A}) \pm \sqrt{(\mathrm{tr}\mathbf{A})^2 - 4(\det \mathbf{A})}}{2}.$$

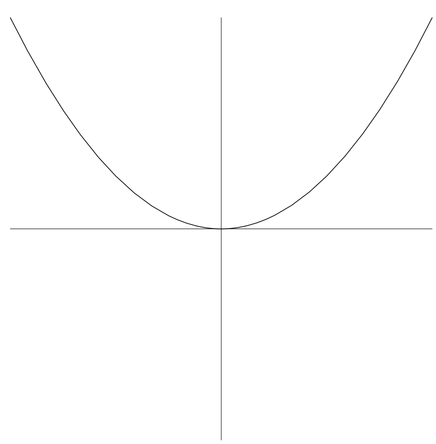
Summary of Phase Portraits

Assume $\det \mathbf{A} \neq 0$. Then zero is not an eigenvalue of $\mathbf{A}$.

1. Real and distinct eigenvalues
   (a) sink
   (b) saddle
   (c) source
2. Complex eigenvalues
   (a) spiral sink
   (b) center
   (c) spiral source
3. Real and repeated eigenvalues
   (a) sink with one eigenline in the phase portrait
   (b) source with one eigenline in the phase portrait
   (c) sink where every solution is a straight-line solution
   (d) source where every solution is a straight-line solution

What if $\det \mathbf{A} = 0$?

We can organize these different types using a plane with unusual coordinate axes.



You can turn on the trace-determinant plane in the `LinearPhasePortraits` tool.

Example. Consider the one-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 1 \\ 0 & d \end{pmatrix} \mathbf{Y}.$$