Eigenvalues, eigenvectors, general solutions, and phase portraits

To find the straight-line solutions of the linear system $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$, we need to find the eigenvalues and associated eigenvectors of the matrix $\mathbf{A}$. We find the eigenvalues using the characteristic equation $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$.

**Example.** Find the general solution to $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}$. 

Using \texttt{HPGSystemSolver}, we plot the phase portrait for this system.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{phaseportrait.png}
\end{figure}

Facts about eigenvalues and eigenvectors: Given a $2 \times 2$ matrix $A$,

1. The characteristic equation can have two real roots, one real root of multiplicity two, or two complex conjugate roots.

2. Given an eigenvector $\mathbf{Y}_0$ associated to an eigenvalue $\lambda$, then any nonzero scalar multiple $c\mathbf{Y}_0$ is also an eigenvector associated to $\lambda$.

3. Eigenvectors associated to distinct eigenvalues are linearly independent.
Summary of Case of Two Distinct Real Eigenvalues

Suppose \( \mathbf{A} \) is a matrix with two eigenvalues \( \lambda_1 \) and \( \lambda_2 \). To be consistent, we will assume that \( \lambda_1 < \lambda_2 \), that \( \mathbf{v}_1 \) is an eigenvector associated to \( \lambda_1 \), and that \( \mathbf{v}_2 \) is an eigenvector associated to \( \lambda_2 \). The general solution of

\[
\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}
\]

is \( \mathbf{Y}(t) = k_1 e^{\lambda_1 t} \mathbf{v}_1 + k_2 e^{\lambda_2 t} \mathbf{v}_2 \).

Case 1: \( \lambda_1 < \lambda_2 < 0 \).
Example. Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}. $$