Linearization

Last class we began to apply what we know about linear systems to nonlinear systems.

**Example.** Consider the van der Pol equation

$$\frac{d^2x}{dt^2} + (x^2 - 1)\frac{dx}{dt} + x = 0.$$

The corresponding system is

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= (1 - x^2)y - x. \end{aligned}$$

We calculated the equilibria and determined that the only equilibrium point is $(0, 0)$, and the linearized system near $(0, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{Y}.$$

**Example.** Consider the (undamped) pendulum

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0.$$

The corresponding system is

$$\begin{aligned} \frac{d\theta}{dt} &= v \\ \frac{dv}{dt} &= -\sin\theta. \end{aligned}$$

There are equilibria at $(\theta, v) = (k\pi, 0)$ for all integers $k$.

The linearized system near $(0, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y},$$

and the linearized system near $(\pi, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{Y},$$

Given the (nonlinear) system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y),$$

its **Jacobian** at the point $(x_0, y_0)$ is the matrix

$$\mathbf{J}(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

and its linearization at $(x_0, y_0)$ is the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}\mathbf{Y}.$$

For the pendulum, we have one linearization for each equilibrium point:

For the van der Pol equation, we obtain the linearization:

**Linearization Theorem** Let $\mathbf{Y}_0$ be an equilibrium point for the nonlinear autonomous system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$$

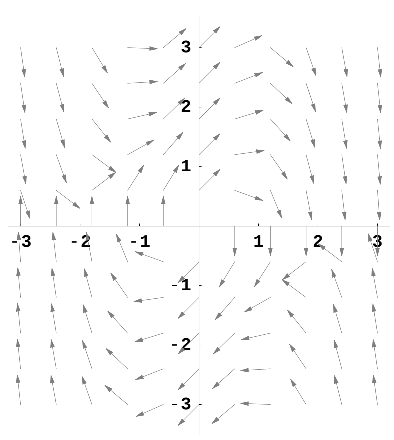
and let

$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}\mathbf{Y}$$

be the corresponding linearized system. If the eigenvalues of $\mathbf{J}$ are not purely imaginary, then the solution curves of the nonlinear system near $\mathbf{Y}_0$ behave in the same qualitative way as the solution curves of the linear system.

**Example.** Consider the van der Pol equation near the origin. The linearized system is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{Y}.$$

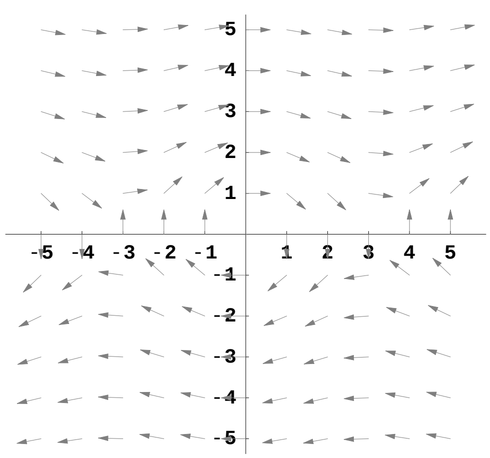


**Example.** Consider the pendulum equation. The linearized system near $(\pi, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{Y}.$$

The linearized system near $(0, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$

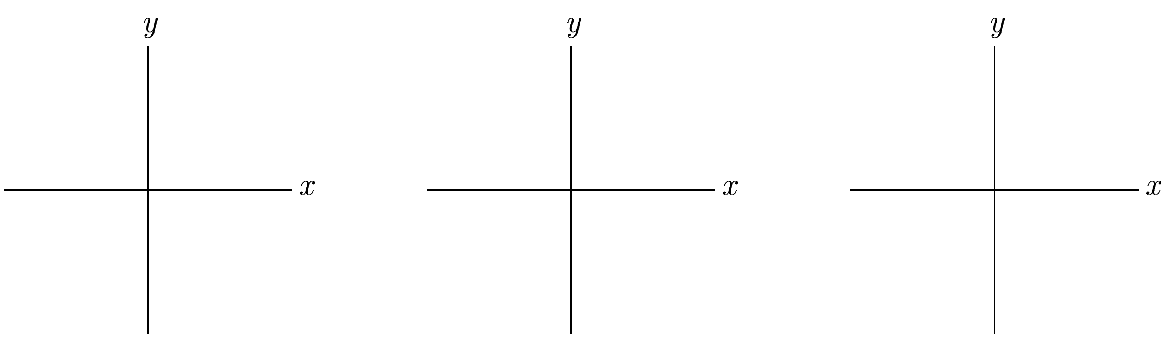


What is special about the case of purely imaginary eigenvalues in the linearization?

**Example.** Consider the one-parameter family of systems

$$\begin{aligned} \frac{dx}{dt} &= -y + \alpha x(x^2 + y^2) \\ \frac{dy}{dt} &= \quad x + \alpha y(x^2 + y^2) \end{aligned}$$

where $\alpha$ is a parameter. Note that $(0, 0)$ is always an equilibrium point.




$\alpha < 0$ $\quad$ $\alpha = 0$ $\quad$ $\alpha > 0$