More on the Laplace transform

Last class we defined the Laplace transform.

Definition. The *Laplace transform* of the function $y(t)$ is the function

$$Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt.$$

This transform is an "operator" (a function on functions). It transforms the function $y(t)$ into the function $Y(s)$.

Notation: We often represent this operator using the script letter $\mathcal{L}$. In other words,

$$\mathcal{L}[y] = Y.$$

For example,

$$\mathcal{L}[1] = \frac{1}{s},$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad \text{and}$$

$$\mathcal{L}[\sin t] = \frac{1}{s^2+1}.$$

Note that even if $y(t)$ is defined for all t, the Laplace transform $Y(s)$ may not be defined for all s.

Properties of the Laplace transform There are two properties of the Laplace transform that make it well suited for solving linear differential equations:

1. $\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$
2. $\mathcal{L}$ is a linear transform.

Example. Solve the initial-value problem

$$\frac{dy}{dt} - 3y = e^{2t}, \quad y(0) = 4.$$

1. Transform both sides of the equation:

2. Solve for $\mathcal{L}[y]$:

3. Calculate the inverse Laplace transform:

Is this the right answer? Do we need Laplace transforms to calculate it?

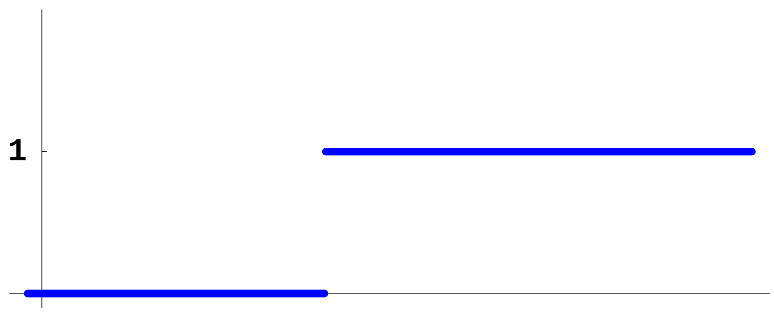
Discontinuous differential equations

The Laplace transform works well on linear differential equations that are discontinuous in one way or another.

Definition. The *Heaviside function* $u_a(t)$ is the function defined by

$$u_a(t) = \begin{cases} 0, & \text{if } t < a; \\ 1, & \text{if } t \geq a. \end{cases}$$

Thus $u_a(t)$ has a discontinuity at $t = a$ where it jumps from 0 to 1. Note that the `step(t)` function in `DETools` is the same function as $u_0(t)$.



Here's how you can use the Heaviside function to avoid piecewise definitions:

Example. Consider $g(t) = 2t + u_1(t)(2 - 2t)$.

Laplace transforms are very convenient if we have discontinuous forcing. Remember the process for solving differential equations using Laplace transforms:

1. Transform both sides of the differential equation.
2. Determine $\mathcal{L}[y]$.
3. Compute the inverse Laplace transform of $\mathcal{L}[y]$.

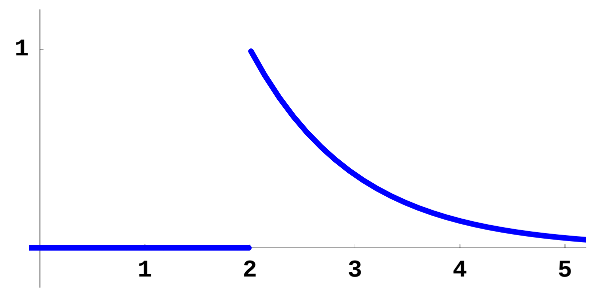
How do we calculate the Laplace transform of a discontinuous function?

Example. Let's calculate $\mathcal{L}[u_a]$ directly from the definition of $\mathcal{L}$.

In order to calculate inverse Laplace transforms, we need another property of the transform.

Rule 3: Shifting the t-axis. $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}\mathcal{L}[f]$.

Example. Calculate $\mathcal{L}[g]$ where $g(t) = u_2(t)\,e^{-(t-2)}$.



Why does the shifting rule work the way that it does?

Shifting the t-axis. Let's compute

$\mathcal{L}[u_a(t)f(t-a)] =$