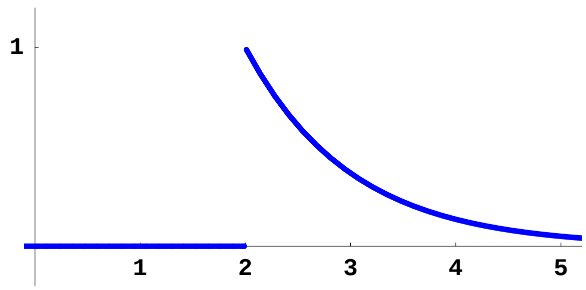
More on Laplace transforms for discontinuous equations

In order to calculate inverse Laplace transforms, we need another property of the transform.

Rule 3: Shifting the t-axis. $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}\mathcal{L}[f]$.

Example. Calculate $\mathcal{L}[g]$ where $g(t) = u_2(t)\, e^{-(t-2)}$.



Why does the shifting rule work the way that it does?

Shifting the t-axis. Let's compute

$\mathcal{L}[u_a(t)f(t-a)] =$

Now let's see how we can use these properties of the Laplace transform to solve an initial-value problem that involves discontinuous forcing.

Example. Solve the initial-value problem

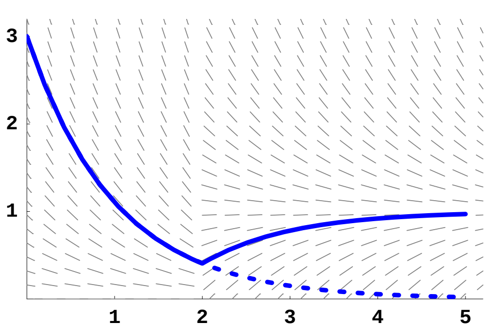
$$\frac{dv}{dt} + v = u_2(t), \quad v(0) = 3.$$

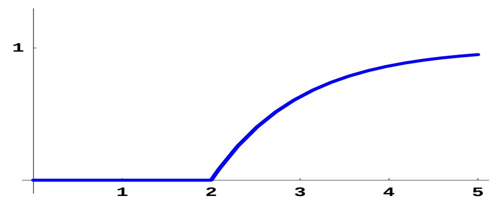
1. Transform both sides of the equation:

2. Solve for $\mathcal{L}[v]$:

3. Calculate the inverse Laplace transform:

Now let's plot the solution to the initial-value problem using `HPGSolver`. The graph of the solution is shown on the left below. The graph on the right is the graph of the function $u_2(t)(1 - e^{-(t-2)})$.





Laplace transforms and second-order equations

So far we have only applied the Laplace transform to first-order equations. Now we consider second-order equations.

Recall the rule for Laplace transforms of derivatives: $\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$. What does this rule say about $\mathcal{L}\left[\frac{d^2y}{dt^2}\right]$?

Now that we have this rule, we also need to add to our table of Laplace transforms. Since sine and cosine often appear as parts of the solutions to second-order equations, let's determine their Laplace transforms.

There are a number of ways to compute these transforms—using integration by parts, using Euler's formula, and even using the fact that sine and cosine are solutions to certain very special second-order equations. *Mathematica* tells us that

$$\mathcal{L}[e^{i\omega t}] = \frac{1}{s - i\omega}.$$

Let's use this fact to determine $\mathcal{L}[\sin \omega t]$ and $\mathcal{L}[\cos \omega t]$.